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Parallel hybrid methods for relatively nonexpansive mappings

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Abstract. We prove strong convergence theorems by the parallel hybrid method proposed by Anh and Chung [2] for a family of relatively nonexpansive mappings. We also deal with another parallel algorithm which is based on the shrinking projection method given by Takahashi, Takeuchi, and Kubota [18].

1. Introduction

In this paper, we study approximation methods for finding a common fixed point of a family of relatively nonexpansive mappings in the sense of Matsushita and Takahashi [13, 14].

One method is a parallel algorithm proposed by Anh and Chung [2], which is called the parallel hybrid method. The parallel hybrid method is a generalization of the hybrid method discussed in Matsushita and Takahashi [14]; see also Nakajo and Takahashi [15].

In Section 3, we prove convergence of the parallel hybrid method for a sequence of quasinonexpansive-like mappings (Theorem 3.2) by using a result established in [7]. Then we obtain a strong convergence theorem for a finite family of relatively nonexpansive mappings, which is a generalization of the main result in [2], as a direct consequence of Theorem 3.2.

We also deal with another approximation method, which is called the parallel shrinking method in Section 4. The parallel shrinking method is based on the shrinking projection method developed by Takahashi, Takeuchi, and Kubota [18]. Using a result obtained in [7], we establish strong convergence of the parallel shrinking method for a sequence of quasinonexpansive-like mappings (Theorem 4.2). Then we show a strong convergence theorem for a finite family of relatively nonexpansive mappings as a direct result of Theorem 4.2.

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2. Preliminaries

Throughout the present paper, E denotes a real Banach space, $\|\cdot\|$ the norm of E, E^* the dual of E, $\langle x, x^*\rangle$ the value of $x^* \in E^*$ at $x \in E$, $\mathbb{N}$ the set of positive integers, and $\mathbb{R}$ the set of real numbers. The norm of E^* is also denoted by $\|\cdot\|$. Strong convergence of a sequence $\{x_n\}$ in E to $x \in E$ is denoted by $x_n \to x$ and weak convergence by $x_n \rightharpoonup x$. The (normalized) *duality mapping* of E is denoted by J, that is, it is a set-valued mapping of E into E^* defined by $Jx = \{x^* \in E^* \colon \langle x, x^*\rangle = \|x\|^2 = \|x^*\|^2\}$ for $x \in E$.

Let U_E denote the unit sphere of E, that is, $U_E = \{x \in E : \|x\| = 1\}$. The norm of E is said to be *Gâteaux differentiable* if the limit

$$\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t} \tag{1}$$

exists for all $x, y \in U_E$. In this case, E is said to be *smooth*. The norm of E is said to be *uniformly Fréchet differentiable* if the limit (1) is attained uniformly for $x, y \in U_E$. In this case, E is said to be *uniformly smooth*. It is known that the duality mapping J is single-valued if E is smooth; J is uniformly norm-to-norm continuous on each bounded subset of E if E is uniformly smooth. A Banach space E is said to be *strictly convex* if $x, y \in U_E$ and $x \neq y$ imply $\|x + y\| < 2$; E is said to be *uniformly convex* if for any $\epsilon > 0$ there exists $\delta > 0$ such that $x, y \in U_E$ and $\|x - y\| \geq \epsilon$ imply $\|x + y\|/2 \leq 1 - \delta$. It is known that E is reflexive and strictly convex if E is uniformly convex; the duality mapping J of E is bijective and J^{-1} is the duality mapping of E^* if E is smooth, strictly convex, and reflexive; see [17] for more details.

Let E be a smooth Banach space. Then we can define a function $\phi\colon E\times E \to \mathbb{R}$ by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2$$

for $x, y \in E$; see [1]. We know that if $\{x_n\}$ and $\{y_n\}$ are bounded sequences in a uniformly convex and uniformly smooth Banach space E, then

$$\|x_n - y_n\| \to 0 \Leftrightarrow \|Jx_n - Jy_n\| \to 0; \tag{2}$$

see, for example, [8, p. 203].

Let E be a smooth, strictly convex, and reflexive Banach space and C a nonempty closed convex subset of E. It is known that for each $x \in E$ there exists a unique point $x_0 \in C$ such that $\phi(x_0, x) = \min\{\phi(y, x) : y \in C\}$. Such a point x_0 is denoted by $\Pi_C(x)$ and Π_C is said to be the *generalized projection* of E onto C; see [1, 12]. We know the following lemma:

LEMMA 2.1. *Let E be a smooth, strictly convex, and reflexive Banach space, C a nonempty closed convex subset of E, $x \in E$, and $z \in C$. Then $z = \Pi_C(x)$ if and only if $\langle y - z, Jx - Jz \rangle \leq 0$ for all $y \in C$.*

Let E be a smooth Banach space, C a nonempty subset of E, and $T\colon C \to E$ a mapping. The set of fixed points of T is denoted by $\mathrm{F}(T)$. A point $p \in C$ is said to be an *asymptotic fixed point* of T [16, 11] if C contains a sequence $\{x_n\}$ such that $x_n \rightharpoonup p$ and $\|x_n - Tx_n\| \to 0$. The set of asymptotic fixed points of T is denoted by $\hat{\mathrm{F}}(T)$. A mapping T is said to be of *type (r)* [7, 8, 6, 5, 9, 3] if $\mathrm{F}(T) \neq \varnothing$ and $\phi(p, Tx) \leq \phi(p, x)$ for all $x \in C$ and $p \in \mathrm{F}(T)$; T is said to be *relatively nonexpansive* [10, 14, 13] if it is of type (r) and $\mathrm{F}(T) = \hat{\mathrm{F}}(T)$. We know that if C is a nonempty closed convex subset of a smooth strictly convex Banach space E and $T\colon C \to E$ is of type (r), then $\mathrm{F}(T)$ is closed and convex; see [14, Proposition 2.4].

Let C be a nonempty subset of a Banach space E, $\{T_n\}$ a sequence of mappings of C into E, and F the set of common fixed points of $\{T_n\}$, that is, $F = \bigcap_{n=1}^{\infty} \mathrm{F}(T_n)$. Suppose that F is nonempty. We say that $\{T_n\}$ satisfies the *condition (Z)* if every weak cluster point of $\{x_n\}$ belongs to F whenever $\{x_n\}$ is a bounded sequence in C such that $\|T_n x_n - x_n\| \to 0$; see [3, 4, 5, 7, 8].

We need the following lemmas and theorems:

LEMMA 2.2. *Let E be a smooth Banach space, C a nonempty subset of E, $U\colon C \to E$ a mapping, N a positive integer, and $S_i\colon C \to E$ a mapping for $i \in \Lambda$, where $\Lambda = \{i \in \mathbb{N}\colon 1 \leq i \leq N\}$. Suppose that $\bigcap_{i \in \Lambda} \mathrm{F}(S_i)$ is nonempty, and that for any $x \in C$ there exists $k \in \arg\max\{\|S_i x - x\| : i \in \Lambda\}$ such that $Ux = S_k x$. Then the following hold:*

(a) *$\|S_i x - x\| \leq \|Ux - x\|$ for all $x \in C$ and $i \in \Lambda$;*

(b) *$\mathrm{F}(U) = \bigcap_{i \in \Lambda} \mathrm{F}(S_i)$;*

(c) *if S_i is of type (r) for every $i \in \Lambda$, then U is of type (r).*

PROOF. Set $F = \bigcap_{i \in \Lambda} \mathrm{F}(S_i)$ and $M(x) = \arg\max\{\|S_i x - x\| : i \in \Lambda\}$ for $x \in C$.

We first show (a). Let $x \in C$ be given. By assumption, there exists $k \in M(x)$ such that $Ux = S_k x$. Thus $\|S_i x - x\| \leq \|S_k x - x\| = \|Ux - x\|$ for every $i \in \Lambda$.

We next show (b). Suppose that $z \in F$. By assumption, there exists $k \in M(z)$ such that $Uz = S_k z$. Since $z \in \mathrm{F}(S_k)$, we have $Uz = z$, and hence $\mathrm{F}(U) \supset$

F. Conversely, suppose that $z \in F(U)$. It follows from (a) that $\|S_i z - z\| \leq \|Uz - z\| = 0$ for every $i \in \Lambda$. Thus $\mathrm{F}(U) \subset F$.

Lastly, we show (c). By assumption and (b), we know that $\mathrm{F}(U) = F \neq \varnothing$. Let $x \in C$ and $z \in \mathrm{F}(U)$ be fixed. Then there exists $k \in M(x)$ such that $Ux = S_k x$. Since $z \in \mathrm{F}(S_k)$ by (b) and S_k is of type (r), we have $\phi(z, Ux) = \phi(z, S_k x) \leq \phi(z, x)$. Thus U is of type (r). □

Lemma 2.3. *Let E be a smooth Banach space, C a nonempty subset of E, $\{U_n\}$ a sequence of mappings of C into E, N a positive integer, and $\{S_{i,n}\}$ a double sequence of mappings of C into E indexed by $(i, n) \in \Lambda \times \mathbb{N}$, where $\Lambda = \{i \in \mathbb{N} \colon 1 \leq i \leq N\}$. Suppose that $\bigcap_{(i,n)\in\Lambda\times\mathbb{N}} \mathrm{F}(S_{i,n})$ is nonempty, and that for any $x \in C$ and $n \in \mathbb{N}$ there exists $k \in \arg\max\{\|S_{i,n}x - x\| : i \in \Lambda\}$ such that $U_n x = S_{k,n}x$. If $\{S_{i,n}\}_{n\in\mathbb{N}}$ satisfies the condition (Z) for every $i \in \Lambda$, then $\{U_n\}$ satisfies the condition (Z).*

Proof. Set $F = \bigcap_{(i,n)\in\Lambda\times\mathbb{N}} \mathrm{F}(S_{i,n})$, $F_n = \bigcap_{i\in\Lambda} \mathrm{F}(S_{i,n})$ for $n \in \mathbb{N}$, $K_i = \bigcap_{n\in\mathbb{N}} \mathrm{F}(S_{i,n})$ for $i \in \Lambda$, and $M(x, n) = \arg\max\{\|S_{i,n}x - x\| : i \in \Lambda\}$ for $x \in C$ and $n \in \mathbb{N}$. It is clear that $K_i \supset F$ for every $i \in \Lambda$. Lemma 2.2 shows that $\mathrm{F}(U_n) = F_n$ for every $n \in \mathbb{N}$. Thus we have $\bigcap_{n\in\mathbb{N}} \mathrm{F}(U_n) = \bigcap_{n\in\mathbb{N}} F_n = F \neq \varnothing$. Let $\{x_n\}$ be a bounded sequence in C such that $\|U_n x_n - x_n\| \to 0$ and $\{x_{n_m}\}$ a subsequence of $\{x_n\}$ such that $x_{n_m} \rightharpoonup z$. Then Lemma 2.2 implies that $\|S_{i,n}x_n - x_n\| \leq \|U_n x_n - x_n\|$ for every $n \in \mathbb{N}$ and $i \in \Lambda$. As a result, $\|S_{i,n}x_n - x_n\| \to 0$ as $n \to \infty$ for every $i \in \Lambda$. Since K_i is nonempty and $\{S_{i,n}\}_{n\in\mathbb{N}}$ satisfies the condition (Z), we see that $z \in K_i$ for every $i \in \Lambda$. Hence Lemma 2.2 also implies that $z \in \bigcap_{i\in\Lambda} K_i = F = \bigcap_{n\in\mathbb{N}} \mathrm{F}(U_n)$. Therefore, $\{U_n\}$ satisfies the condition (Z). □

Lemma 2.4. *Let E be a uniformly convex and uniformly smooth Banach space, C a nonempty subset of E, $T \colon C \to E$ a mapping of type (r), $\{\lambda_n\}$ a sequence in $[0, 1)$, and $S_n \colon C \to E$ a mapping defined by*

$$S_n = J^{-1}[\lambda_n J + (1 - \lambda_n)JT]$$

for $n \in \mathbb{N}$. Then S_n is of type (r) for every $n \in \mathbb{N}$. Moreover, if T is relatively nonexpansive and $\sup_n \lambda_n < 1$, then $\{S_n\}$ satisfies the condition (Z).

Proof. It is clear that $\mathrm{F}(S_n) = \mathrm{F}(T) \neq \varnothing$ for every $n \in \mathbb{N}$ and hence $\bigcap_{n\in\mathbb{N}} \mathrm{F}(S_n) = \mathrm{F}(T) \neq \varnothing$. Let $n \in \mathbb{N}$, $z \in \mathrm{F}(S_n)$, and $x \in C$ be fixed. Since $z \in \mathrm{F}(T)$ and T is of type (r), we have

$$\phi(z, S_n x) \leq \lambda_n \phi(z, x) + (1 - \lambda_n)\phi(z, Tx) \leq \phi(z, x).$$

Therefore, S_n is of type (r) for every $n \in \mathbb{N}$. Next suppose that T is relatively nonexpansive and $\sup_n \lambda_n < 1$. Let $\{x_n\}$ be a bounded sequence in C such that $\|x_n - S_n x_n\| \to 0$ and $\{x_{n_m}\}$ a subsequence of $\{x_n\}$ such that $x_{n_m} \rightharpoonup w$. By assumption, it follows that

$$(1 - \sup_n \lambda_n) \|Jx_n - JTx_n\| \leq \|Jx_n - JS_n x_n\|$$

for every $n \in \mathbb{N}$. Taking into account $\sup_n \lambda_n < 1$ and (2), we conclude that $\|x_n - Tx_n\| \to 0$. Since T is relatively nonexpansive, $w \in \hat{\mathrm{F}}(T) = \mathrm{F}(T) = \bigcap_{n\in\mathbb{N}} \mathrm{F}(S_n)$. Therefore, $\{S_n\}$ satisfies the condition (Z). $\square$

Using [7, Theorems 4.2 and 4.4], [3, Proposition 6], and [8, Lemma 4.3], we obtain the following theorems:

Theorem 2.5. *Let E be a uniformly convex and smooth Banach space, C a nonempty closed convex subset of E, $\{U_n\}$ a sequence of mappings of C into E, and F the set of common fixed points of $\{U_n\}$. Suppose that U_n is of type (r) for every $n \in \mathbb{N}$, F is nonempty, and $\{U_n\}$ satisfies the condition (Z). Let u be a point in E and $\{x_n\}$ a sequence in C defined by $x_1 = \Pi_C(u)$ and*

$$\begin{cases} H_n = \{z \in C \colon \phi(z, U_n x_n) \leq \phi(z, x_n)\}; \\ W_n = \{z \in C \colon \langle z - x_n, Ju - Jx_n \rangle \leq 0\}; \\ x_{n+1} = \Pi_{H_n \cap W_n}(u) \end{cases}$$

for $n \in \mathbb{N}$. Then $\{x_n\}$ converges strongly to $\Pi_F(u)$.

Theorem 2.6. *Let E, C, $\{U_n\}$, F, and u be the same as in Theorem 2.5 and $\{x_n\}$ a sequence in C defined by $x_1 = \Pi_C(u)$, $C_1 = C$, and*

$$\begin{cases} C_{n+1} = \{z \in C \colon \phi(z, U_n x_n) \leq \phi(z, x_n)\} \cap C_n; \\ x_{n+1} = \Pi_{C_{n+1}}(u) \end{cases}$$

for $n \in \mathbb{N}$. Then $\{x_n\}$ converges strongly to $\Pi_F(u)$.

3. Convergence theorems by the parallel hybrid method

In this section, we prove strong convergence of the parallel hybrid method for a sequence of mappings of type (r). Then, applying the convergence result, we obtain a strong convergence theorem for a finite family of relatively nonexpansive mappings, which is a generalization of [2, Theorem 3.1].

In what follows, we assume that E is a smooth, strictly convex, and reflexive Banach space, C is a nonempty closed convex subset of E, u is a point in E, N is a positive integer, and $S_{i,n} \colon C \to E$ is a mapping of type (r) for $i \in \Lambda$ and $n \in \mathbb{N}$, where $\Lambda = \{i \in \mathbb{N} \colon 1 \leq i \leq N\}$. We also assume that F is the set of common fixed points of $\{S_{i,n}\}_{(i,n)\in\Lambda\times\mathbb{N}}$, that is, $F = \bigcap_{(i,n)\in\Lambda\times\mathbb{N}} \mathrm{F}(S_{i,n})$. Under these assumptions, we investigate convergence of a sequence $\{x_n\}$ defined by $x_1 = \Pi_C(u)$ and

$$\begin{cases} i_n \in \arg\max\{\|S_{i,n}x_n - x_n\| : i \in \Lambda\}; \\ H_n = \{z \in C \colon \phi(z, S_{i_n,n}x_n) \leq \phi(z, x_n)\}; \\ W_n = \{z \in C \colon \langle z - x_n, Ju - Jx_n\rangle \leq 0\}; \\ x_{n+1} = \Pi_{H_n \cap W_n}(u) \end{cases}$$

for $n \in \mathbb{N}$.

We first show that the sequence $\{x_n\}$ is well-defined.

Lemma 3.1. *Suppose that F is nonempty. Then $H_n \cap W_n$ is nonempty, closed, and convex for every $n \in \mathbb{N}$, and therefore, $\{x_n\}$ is well-defined.*

Proof. It is clear from the definition that $H_n \cap W_n$ is closed and convex for every $n \in \mathbb{N}$. Thus it is enough to show that $H_n \cap W_n$ is nonempty for every $n \in \mathbb{N}$. Since each $S_{i_n,n}$ is of type (r) and $F \subset \mathrm{F}(S_{i_n,n})$, it follows that $F \subset H_n$ for every $n \in \mathbb{N}$. Lemma 2.1 shows that $W_1 = C$. Hence we have $H_1 \cap W_1 \supset F \cap C = F \neq \varnothing$. We next suppose that there exists $n \in \mathbb{N}$ such that $H_k \cap W_k \neq \varnothing$ for every $k \in \{1, \dots, n\}$. Then $x_1, \dots, x_{n+1}$ are well-defined. Since $x_{k+1} = \Pi_{H_k \cap W_k}(u)$, Lemma 2.1 implies that $H_k \cap W_k \subset W_{k+1}$ for every $k \in \{1, \dots, n\}$. Hence we deduce that

$$\begin{aligned} H_{n+1} \cap W_{n+1} &\supset H_{n+1} \cap (H_n \cap W_n) \supset \cdots \\ &\supset \bigcap_{k=1}^{n+1} H_k \cap W_1 = \bigcap_{k=1}^{n+1} H_k \supset F \neq \varnothing. \end{aligned}$$

Therefore, by induction on n, we conclude that $H_n \cap W_n$ is nonempty for every $n \in \mathbb{N}$. □

Using Theorem 2.5, Lemmas 2.2, 2.3, and 3.1, we can show convergence of $\{x_n\}$.

Theorem 3.2. *Suppose, in addition to the assumptions above, that E is uniformly convex, F is nonempty, and $\{S_{i,n}\}_{n\in\mathbb{N}}$ satisfies the condition (Z) for every*

$i \in \Lambda$. Then $\{x_n\}$ converges strongly to $\Pi_F(u)$.

PROOF. Lemma 3.1 shows that $\{x_n\}$ and $\{i_n\}$ are well-defined. Set

$$M(x,n) = \arg\max\{\|S_{i,n}x - x\| : i \in \Lambda\} \tag{3}$$

for $x \in C$ and $n \in \mathbb{N}$. Let $U_n\colon C \to E$ be a mapping defined by

$$U_n x = \begin{cases} S_{i_n,n}x_n & \text{if } x = x_n; \\ S_{j,n}x & \text{otherwise} \end{cases} \tag{4}$$

for $x \in C$ and $n \in \mathbb{N}$, where j is the minimum element of the set $M(x,n)$. We also know that for any $x \in C$ and $n \in \mathbb{N}$ there exists $k \in M(x,n)$ such that $U_n x = S_{k,n}x$. Lemmas 2.2 and 2.3 imply that $\{U_n\}$ is a sequence of mappings of type (r), $\bigcap_{n\in\mathbb{N}} \mathrm{F}(U_n) = F$, $\bigcap_{n\in\mathbb{N}} \mathrm{F}(U_n)$ is nonempty, and $\{U_n\}$ satisfies the condition (Z). From the definition of U_n, it is clear that $U_n x_n = S_{i_n,n}x_n$ for every $n \in \mathbb{N}$. Using Theorem 2.5, we conclude that $\{x_n\}$ converges strongly to $\Pi_F(u)$. □

Applying Theorem 3.2 and Lemma 2.4, we can prove the following theorem:

THEOREM 3.3. *Let E be a uniformly convex and uniformly smooth Banach space, C a nonempty closed convex subset of E, N a positive integer, $\{\alpha_n^i\}$ a double sequence in $[0,1)$ indexed by $n \in \mathbb{N}$ and $i \in \Lambda$, and $T_i\colon C \to E$ a relatively nonexpansive mapping for $i \in \Lambda$, where $\Lambda = \{i \in \mathbb{N}\colon 1 \le i \le N\}$. Suppose that $\sup_n \alpha_n^i < 1$ for every $i \in \Lambda$ and K is nonempty, where $K = \bigcap_{i\in\Lambda} \mathrm{F}(T_i)$. Let u be a point in E and $\{x_n\}$ a sequence in C defined by $x_1 = \Pi_C(u)$ and*

$$\begin{cases} i_n \in \arg\max\left\{\left\|J^{-1}\left[\alpha_n^i Jx_n + (1-\alpha_n^i)JT_i x_n\right] - x_n\right\| : i \in \Lambda\right\}; \\ y_n = J^{-1}\left[\alpha_n^{i_n} Jx_n + (1-\alpha_n^{i_n})JT_{i_n}x_n\right]; \\ H_n = \{z \in C\colon \phi(z,y_n) \le \phi(z,x_n)\}; \\ W_n = \{z \in C\colon \langle z - x_n, Ju - Jx_n\rangle \le 0\}; \\ x_{n+1} = \Pi_{H_n\cap W_n}(u) \end{cases}$$

for $n \in \mathbb{N}$. Then $\{x_n\}$ converges strongly to $\Pi_K(u)$.

PROOF. Let $S_{i,n}\colon C \to E$ be a mapping defined by

$$S_{i,n} = J^{-1}\left[\alpha_n^i J + (1-\alpha_n^i)JT_i\right]$$

for $n \in \mathbb{N}$ and $i \in \Lambda$. Then it is clear that $\mathrm{F}(S_{i,n}) = \mathrm{F}(T_i)$ for every $n \in \mathbb{N}$ and

$i \in \Lambda$, and hence

$$\bigcap_{(i,n)\in\Lambda\times\mathbb{N}} \mathrm{F}(S_{i,n}) = \bigcap_{i\in\Lambda} \mathrm{F}(T_i) = K \neq \varnothing.$$

Lemma 2.4 shows that each $S_{i,n}$ is of type (r) and $\{S_{i,n}\}_{n\in\mathbb{N}}$ satisfies the condition (Z) for every $i \in \Lambda$. By definition, we see that

$$i_n \in \arg\max\{\|S_{i,n}x_n - x_n\| : i \in \Lambda\} \text{ and } y_n = S_{i_n,n}x_n$$

for every $n \in \mathbb{N}$. Therefore Theorem 3.2 implies the conclusion. □

Theorem 3.3 is a slight generalization of the main result in [2]. Indeed, in [2, Theorem 3.1], it is assumed that T_i is continuous and $\alpha_n^1 = \alpha_n^i$ for every $i \in \Lambda$, and that $\alpha_n^1 \to 0$ as $n \to \infty$.

4. Convergence theorems by the parallel shrinking method

In this section, we prove strong convergence of the parallel shrinking method for a sequence of mappings of type (r). Then, applying the convergence result, we obtain a strong convergence theorem for a finite family of relatively nonexpansive mappings.

In what follows, we assume that E, C, u, N, $S_{i,n}$, Λ, and F are the same as in Section 3. We investigate convergence of a sequence $\{x_n\}$ defined by $x_1 = \Pi_C(u)$, $C_1 = C$, and

$$\begin{cases} i_n \in \arg\max\{\|S_{i,n}x_n - x_n\| : i \in \Lambda\}; \\ C_{n+1} = \{z \in C \colon \phi(z, S_{i_n,n}x_n) \leq \phi(z, x_n)\} \cap C_n; \\ x_{n+1} = \Pi_{C_{n+1}}(u) \end{cases}$$

for $n \in \mathbb{N}$.

We first show that the sequence $\{x_n\}$ is well-defined.

Lemma 4.1. *Suppose that F is nonempty. Then C_n is nonempty, closed, and convex for every $n \in \mathbb{N}$, and therefore, $\{x_n\}$ is well-defined.*

Proof. It is clear from the definition that C_n is closed and convex for every $n \in \mathbb{N}$. Thus it is enough to show that $C_n \supset F$ holds for every $n \in \mathbb{N}$. It clear that $C_1 = C \supset F$. Suppose that $C_n \supset F$ for some $n \in \mathbb{N}$. Since each $S_{i_n,n}$ is of type (r) and $\mathrm{F}(S_{i_n,n}) \supset F$, it follows that $C_{n+1} \supset F \cap C_n \supset F$. Therefore, by induction on

n, we conclude that $C_n \supset F$ for every $n \in \mathbb{N}$. □

Using Theorem 2.6, Lemmas 2.2, 2.3, and 4.1, we can show convergence of $\{x_n\}$.

THEOREM 4.2. *Suppose, in addition to the assumptions above, that E is uniformly convex, F is nonempty, and $\{S_{i,n}\}_{n\in\mathbb{N}}$ satisfies the condition (Z) for every $i \in \Lambda$. Then $\{x_n\}$ converges strongly to $\Pi_F(u)$.*

PROOF. Lemma 4.1 shows that $\{x_n\}$ and $\{i_n\}$ are well-defined. Let $M(x, n)$ be a subset of Λ defined by (3) and $U_n : C \to E$ a mapping by (4) for $x \in C$ and $n \in \mathbb{N}$. We know that for any $x \in C$ and $n \in \mathbb{N}$ there exists $k \in M(x, n)$ such that $U_n x = S_{k,n} x$. Lemmas 2.2 and 2.3 imply that $\{U_n\}$ is a sequence of mappings of type (r), $\bigcap_{n\in\mathbb{N}} \mathrm{F}(U_n) = F$, $\bigcap_{n\in\mathbb{N}} \mathrm{F}(U_n)$ is nonempty, and $\{U_n\}$ satisfies the condition (Z). From the definition of U_n, it is clear that $U_n x_n = S_{i_n,n} x_n$ for every $n \in \mathbb{N}$. Therefore, by Theorem 2.6, we conclude that $\{x_n\}$ converges strongly to $\Pi_F(u)$. □

Applying Theorem 4.2 and Lemma 2.4, we can obtain the following theorem. We omit the proof because it is practically the same as that of Theorem 3.3.

THEOREM 4.3. *Let E be a uniformly convex and uniformly smooth Banach space, C a nonempty closed convex subset of E, N a positive integer, $\{\alpha_n^i\}$ a double sequence in $[0, 1)$ indexed by $n \in \mathbb{N}$ and $i \in \Lambda$, and $T_i : C \to E$ a relatively nonexpansive mapping for $i \in \Lambda$, where $\Lambda = \{i \in \mathbb{N} : 1 \le i \le N\}$. Suppose that $\sup_n \alpha_n^i < 1$ for every $i \in \Lambda$ and K is nonempty, where $K = \bigcap_{i\in\Lambda} \mathrm{F}(T_i)$. Let u be a point in E and $\{x_n\}$ a sequence in C defined by $x_1 = \Pi_C(u)$, $C_1 = C$, and*

$$\begin{cases} i_n \in \arg\max \left\{ \left\| J^{-1}\left[\alpha_n^i J x_n + (1-\alpha_n^i) J T_i x_n\right] - x_n \right\| : i \in \Lambda \right\}; \\ y_n = J^{-1}\left[\alpha_n^{i_n} J x_n + (1-\alpha_n^{i_n}) J T_{i_n} x_n\right]; \\ C_{n+1} = \{z \in C : \phi(z, y_n) \le \phi(z, x_n)\} \cap C_n; \\ x_{n+1} = \Pi_{C_{n+1}}(u) \end{cases}$$

for $n \in \mathbb{N}$. Then $\{x_n\}$ converges strongly to $\Pi_K(u)$.

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