A NOTE ON KNOTS WITH H(2)-UNKNOTTING NUMBER ONE

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A NOTE ON KNOTS WITH H(2)-UNKNOTTING NUMBER ONE

YUANYUAN BAO

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Abstract

We give an obstruction to unknotting a knot by adding a twisted band, derived from Heegaard Floer homology.

1. Introduction

Many unknotting operations have been defined and studied in knot theory. For example, as well-known, (a), (b) (cf. [8, 10]) and (c) in Fig. 1 are three types of unknotting operations. Especially, (c) was introduced by Hoste, Nakanishi and Taniyama [4], which they called H(n)-move. Here n is the number of arcs inside the circle. Note that an H(n)-move is required to preserve the component number of the diagram. The H(n)-unknotting number of a knot is the minimal number of H(n)-moves needed to change the knot into the unknot. In this note, we focus on the special case when n equals two. Given two knots \( K \) and \( K' \), when \( K' \) is obtained from \( K \) by applying an H(2)-move, we also alternatively say that \( K' \) is obtained from \( K \) by adding a twisted band, as shown in Fig. 2. Following [4], we denote the H(2)-unknotting number of a knot \( K \) by \( u_2(K) \). In this note, we give a necessary condition for a knot \( K \) to have \( u_2(K) = 1 \), by using a method introduced by Ozsváth and Szabó [15].

The question whether a given knot has H(2)-unknotting number one should be traced back to Riley. He made the conjecture that the figure-eight knot could never be unknotted by adding a twisted band. Lickorish confirmed this conjecture in [7]. Here we give a brief review of his method. Given a knot \( K \), let \( \Sigma(K) \) denote the double-branched cover of \( S^3 \) along \( K \) and let \( \lambda: H_1(\Sigma(K), \mathbb{Z}) \times H_1(\Sigma(K), \mathbb{Z}) \to \mathbb{Q}/\mathbb{Z} \) be the linking form of \( \Sigma(K) \). Lickorish proved that if the knot \( K \) can be unknotted by adding a twisted band, then \( H_1(\Sigma(K), \mathbb{Z}) \) is cyclic and it has a generator \( g \) such that \( \lambda(g, g) = \pm1/\det(K) \), where \( \det(K) \) is the determinant of \( K \). For the figure-eight knot \( 4_1 \), the linking form has the form \( \lambda(g, g) = 2/5 \) for some generator \( g \in H_1(\Sigma(4_1)) \cong \mathbb{Z}/5\mathbb{Z} \). If there is another generator \( g' = xg \) such that \( \lambda(g', g') = \pm1/5 \), we have \( 2x^2 \equiv \pm1 \) (mod 5), while there is no such an integer \( x \) satisfying the condition. Therefore Riley’s conjecture holds.

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Fig. 1. Some unknotting operations.

Fig. 2. Adding a twisted band to a knot diagram.
Now we turn to the description of our result. Consider a negative-definite symmetric \( n \times n \) matrix \( Q \) over \( \mathbb{Z} \), and suppose \( |\text{det}(Q)| \) is \( p \). Then define a group

\[
G_Q := \mathbb{Z}^n / \text{Im}(Q).
\]

A \textit{characteristic vector} for \( Q \) is an element in

\[
\text{char}(Q) = \{ \xi = (\xi_1, \xi_2, \ldots, \xi_n) \in \mathbb{Z}^n \mid \xi^t v \equiv v^t Q v \pmod{2} \text{ for any } v \in \mathbb{Z}^n \} = \{ \xi \in \mathbb{Z}^n \mid \xi_i \equiv Q_{ii} \pmod{2} \text{ for } 1 \leq i \leq n \}.
\]

Suppose \( p \) is odd, and consider the map (cf. [12, 15])

\[
M_Q : G_Q \to \mathbb{Q}
\]

defined by

\[
M_Q(\alpha) = \max \left\{ \frac{\xi^t Q^{-1} \xi + n}{4} \middle| \xi \in \text{char}(Q), [\xi] = \alpha \in G_Q \right\}.
\]

Now we recall the definition of Goeritz matrix. Given a knot diagram, color this diagram in checkerboard fashion such that the unbounded region has black color. Let \( f_0, f_1, \ldots, f_k \) denote the black regions and \( f_0 \) correspond to the unbounded one. Define the sign of a crossing as in Fig. 3. Then the Goeritz matrix \( A \) is the \( k \times k \) symmetric matrix defined as follows,

\[
q_{ij} = \begin{cases} 
\text{the signed count of crossings adjacent to } f_i & \text{if } i = j, \\
-\text{the signed count of crossings joining } f_i \text{ and } f_j & \text{if } i \neq j
\end{cases}
\]

for \( i, j = 1, 2, \ldots, k \).

Our result about \( H(2) \)-unknotting number is as follows:
Theorem 1.1. Let $K$ be an alternating knot with $|\det K| = p$, and let $A$ be the negative-definite Goeritz matrix corresponding to a reduced alternating diagram of $K$ or its mirror image. Since $K$ is a knot, we see that $p$ is an odd number. Suppose $G_A$ is the group presented by $A$. If $u_2(K) = 1$, then there is an isomorphism $\phi: \mathbb{Z}/p\mathbb{Z} \to G_A$ and a sign $\epsilon \in \{+1, -1\}$ with the properties that for all $i \in \mathbb{Z}/p\mathbb{Z}$:

$$I_{\phi, \epsilon}(i) := \epsilon \cdot M_A(\phi(i)) + \frac{1}{4} \left( \frac{1}{p} \left( \frac{p + (-1)^i p}{2} - i \right)^2 - 1 \right) = 0 \pmod{2},$$

and

$$I_{\phi, \epsilon}(i) \geq 0.$$

Here we abuse $i$ to denote both the element in $\mathbb{Z}/p\mathbb{Z}$ and its representative in the set $\{0, 1, 2, \ldots, p - 1\}$.

If one is familiar with the work in [15], the proof is immediate. We will give the proof in Section 2.

The $H(2)$-unknotting number of a knot is an interesting knot invariant. It is closely related to the 3-dimensional and 4-dimensional crosscap numbers of a knot. It can be defined in some different viewpoints, as indicated by Taniyama and Yasuhara [17]. Many researches concerning it can be found in [18, 6, 1] and other papers.

In order to check that Theorem 1.1 works better in some cases than the existing criteria, we post the knot $P(13, 4, 11)$ as an example. We determine that it has $H(2)$-unknotting number 2, which cannot seem to be detected by the other methods that the author knows.

Corollary 1.2. The pretzel knot $P(13, 4, 11)$ has $H(2)$-unknotting number 2.

2. Proofs

2.1. Preliminaries. Almost all the ingredients contained in this subsection can be found in [15], or an earlier paper [13]. But for intactness, we include them here.

If $X$ is an oriented 3- or 4-manifold, the second cohomology $H^2(X, \mathbb{Z})$ acts on the set of spin$^c$-structures $\text{Spin}^c(X)$ freely and transitively. Each spin$^c$-structure $s \in \text{Spin}^c(X)$ has the first Chern class $c_1(s) \in H^2(X, \mathbb{Z})$, and the relation to the action is $c_1(s + h) = c_1(s) + 2h$ for any $h \in H^2(X, \mathbb{Z})$.

Let $Y$ be an oriented rational homology 3-sphere and $s$ be a spin$^c$-structure over $Y$. Then there is Heegaard Floer homology associated with the pair $(Y, s)$. In this note, we use Heegaard Floer homology with coefficients in the field $\mathbb{F} := \mathbb{Z}/2\mathbb{Z}$. There are several versions of this homology. One version is $HF^+(Y, s)$, which is a $\mathbb{Q}$-graded
module over the polynomial algebra \( \mathbb{F}[U] \). That is

\[
HF^+(Y, s) = \bigoplus_{i \in \mathbb{Q}} HF_i^+(Y, s),
\]

where multiplication by \( U \) lowers the grading by two. In each grading \( i \in \mathbb{Q} \), \( HF_i^+(Y, s) \) is a finite-dimensional \( \mathbb{F} \)-vector space. A simpler version is \( HF^\infty(Y) \), and it satisfies \( HF^\infty(Y, s) = \mathbb{F}[U, U^{-1}] \) for each \( s \in \text{Spin}^c(Y) \) [14, Theorem 10.1]. It is also \( \mathbb{Q} \)-graded and multiplication by \( U \) lowers its grading by two.

For any spin\(^c\)-structure \( s \), there is a natural \( \mathbb{F}[U] \)-equivariant map

\[
\pi : HF^\infty(Y, s) \to HF^+(Y, s),
\]

which preserves the \( \mathbb{Q} \)-grading. We use \( \pi_i \) to denote the restriction of \( \pi \) on the grading \( i \). Then \( \pi_i \) is zero for all sufficiently negative gradings and an isomorphism in all sufficiently positive gradings. Ozsváth and Szabó defined an invariant \( d(Y, s) \) from the map \( \pi \), which is called the correction term of the pair \( (Y, s) \). Precisely, we have

\[
d(Y, s) := \min \{ i \in \mathbb{Q} \mid \pi_i \text{ is non-zero} \}.
\]

The correction terms for \( Y \) and \( -Y \), where "-" means the reversion of orientation, are related by the formula

\[
d(-Y, s) = -d(Y, s)
\]

under the natural identification \( \text{Spin}^c(Y) \cong \text{Spin}^c(-Y) \).

The map \( \pi \) behaves naturally under cobordisms. Let \( Y_1 \) and \( Y_2 \) be two oriented rational homology 3-spheres. We say a smooth connected oriented 4-manifold \( X \) is a cobordism from \( Y_1 \) to \( Y_2 \) if the boundary of \( X \) is given by \( \partial X = -Y_1 \cup Y_2 \). Suppose \( X \) is a cobordism from \( Y_1 \) to \( Y_2 \) and \( t \) is a spin\(^c\)-structure of \( X \). Then there is a homomorphism

\[
F^\alpha_{X, t} : HF^\alpha(Y_1, s_1) \to HF^\alpha(Y_2, s_2),
\]

where \( HF^\alpha \) denotes any version of Heegaard Floer homology and \( s_i \) is the restriction of \( t \) to \( Y_i \) for \( i = 1, 2 \) (we simply express it as \( s_i = t|_{Y_i} \)). The map \( \pi \) and the map \( F^\alpha_{X, t} \) fit into the following commutative diagram:

\[
\begin{array}{ccc}
HF^\infty(Y_1, s_1) & \xrightarrow{F^\infty_{X, t}} & HF^\infty(Y_2, s_2) \\
\pi \downarrow & & \pi \downarrow \\
HF^+(Y_1, s_1) & \xrightarrow{F^+_X} & HF^+(Y_2, s_2).
\end{array}
\]
If $X$ is a negative-definite cobordism, the proof of Theorem 9.1 in [13] (also mentioned in the proof of [13, Proposition 9.9]) tells us that $F_{X,t}^{\infty}$ is an isomorphism.

Suppose that $Y$ is an oriented rational homology 3-sphere, that $X$ is a negative-definite simply connected 4-manifold with $\partial X = Y$ and that $t \in \text{Spin}^c(X)$. Then it is shown in [13] that

\[(1) \quad d(Y, t|_Y) \geq \frac{c_1^2(t) + b_2(X)}{4},\]
\[(2) \quad d(Y, t|_Y) = \frac{c_1^2(t) + b_2(X)}{4} \quad \text{(mod 2)}.\]

Here (1) follows directly from [13, Theorem 9.6], while (2) is not clearly written. For readers’ convenience, we explain it here. Consider $X$ minus a point as a cobordism $W$ from $S^3$ to $Y$. Then we have the following commutative diagram

\[
\begin{array}{ccc}
HF^{\infty}(S^3, t|_{S^3}) & \xrightarrow{F_{W,t}^{\infty}} & HF^{\infty}(Y, t|_Y) \\
\pi \downarrow & & \downarrow \pi \\
HF^{+}(S^3, t|_{S^3}) & \xrightarrow{F_{W,t}^{+}} & HF^{+}(Y, t|_Y),
\end{array}
\]

and $F_{W,t}^{\infty}$ is an isomorphism. There is an element $\xi \in HF^{\infty}(Y, t|_Y)$ with the property that its $\mathbb{Q}$-grading $\text{gr}(\xi)$ is $d(Y, t|_Y)$. Suppose the preimage of $\xi$ in $HF^{\infty}(S^3, t|_{S^3})$ is $\eta$. Then we have

\[d(Y, t|_Y) - \text{gr}(\eta) = \text{gr}(\xi) - \text{gr}(\eta) = \frac{c_1^2(t) - 2\chi(W) - 3\sigma(W)}{4} = \frac{c_1^2(t) + b_2(X)}{4}.\]

The first equality follows from our choice of $\xi$, the second one follows from Equation (4) in [13], and the last one holds because of the fact that $2\chi(W) + 3\sigma(W) + b_2(X) = 0$. Precisely we have

\[2\chi(W) + 3\sigma(W) + b_2(X) = 2(b_0(W) - b_1(W) + b_2(W) - b_3(W) + b_4(W)) - 3b_2(W) + b_2(W)\]
\[= 2(b_0(W) - b_1(W) - b_3(W) + b_4(W))\]
\[= 2(b_0(W) - 2b_1(W) - 1 + b_4(W)) = 0.\]

Here $b_i(W)$ denotes the $i$-th Betti number of $W$. The first equality comes from our assumption that $X$ is negative-definite. The third equality follows from the fact that $b_3(W) = b_1(W) + 1$, obtained from the relation $H_3(W) \cong H_3(W, S^3 \cup Y) \oplus \mathbb{Z}$, Poincaré duality and the universal coefficient theorem. The last equality comes from the facts that $b_0(W) = 1$ and $b_4(W) = 0$, and our assumption that $X$ is simply connected. For
the 3-sphere $S^3$, as an $\mathbb{F}$-vector space, we know that ([14, Theorem 10.1])

$$HF^\infty(S^3, t|_{S^3}) = \bigoplus_{i=-\infty}^{\infty} \mathbb{F}(2i),$$

where $\mathbb{F}(j)$ denotes the summand supported on grading $j$. Therefore we see that $\text{gr}(\eta) = 0 \pmod{2}$. Now (2) follows.

Remember that $d(S^3, t|_{S^3}) = 0$ and that $HF^\infty(S^3, t|_{S^3}) = \mathbb{F}[U, U^{-1}]$, and therefore we obtain $\text{gr}(\eta) = 0 \pmod{2}$. Now (2) follows obviously.

Suppose further for simplicity that $X$ is simply-connected and that the order of $H^2(Y, \mathbb{Z})$ is odd. Then there exists a group structure on the space $\text{Spin}^c(Y)$ by identifying $s \in \text{Spin}^c(Y)$ with $c_1(s) \in H^2(Y, \mathbb{Z})$. In the following, we denote the correction term $d(Y, s)$ by $d(Y, c_1(s))$ if necessary. Let $r$ denote the second Betti number of $X$.

Then we have the following exact sequence:

$$0 \to H_2(X) = \mathbb{Z}^r \xrightarrow{\tau} H^2(X) = \mathbb{Z}^r \xrightarrow{j^*} H^2(Y) \to H_1(X) = 0.$$

Fix a basis for $H_2(X)$ and let $B$ be the matrix of the intersection form of $X$. Then $B$ is a symmetric negative-definite $r \times r$ integer matrix with $|\det B| = |H^2(Y, \mathbb{Z})|$. A $\text{Spin}^c$-structure $s \in \text{Spin}^c(Y)$ is the restriction of a $\text{spin}^c$-structure $t \in \text{Spin}^c(X)$ on $Y$ if and only if $j^*(c_1(t)) = c_1(s)$.

In fact, the map $\tau$ under the given basis of $H_2(X)$ is presented by the matrix $B$. We define $\varphi$ as the map $\text{Coker}(\tau) = G_B \xrightarrow{j^*} H^2(Y)$, where $j^*$ is the map induced from $j^*$ on the cokernel of $\tau$. It is obvious from the exact sequence that $\varphi$ is an isomorphism. Under $\varphi$ the set of characteristic vectors $\text{char}(B)$ is equal to the set $\{c_1(t) \mid t \in \text{Spin}^c(X)\} \subset H^2(X, \mathbb{Z})$. If we suppose the first Chern class $c_1(t)$ corresponds to the characteristic vector $\xi$, then $c_1(t) = \xi^t B^{-1} \xi$.

Under these identifications, (1) and (2) can be written as follows. For any $s \in \text{Spin}^c(Y)$ and any $\xi \in \text{char}(B)$ with $c_1(s) = \varphi([\xi])$, there are

$$d(Y, c_1(s)) \geq \frac{\xi^t B^{-1} \xi + r}{4}$$

and

$$d(Y, c_1(s)) = \frac{\xi^t B^{-1} \xi + r}{4} \pmod{2}.$$

This is equivalent to say under the isomorphism $\varphi: G_B \to H^2(Y, \mathbb{Z})$ the following hold for any $\alpha \in G_B$:

$$d(Y, \varphi(\alpha)) \geq M_B(\alpha),$$

$$d(Y, \varphi(\alpha)) = M_B(\alpha) \pmod{2}. 
 \quad (3)$$
2.2. Proof of Theorem 1.1. When $K$ is an alternating knot in $S^3$, the correction terms for $\Sigma(K)$ have an extremely easy combinatorial description as follows.

**Theorem 2.1** (Ozsváth–Szabó [15, 16]). If $K$ is an alternating knot and $A$ denotes a Goeritz matrix associated to a reduced alternating projection of $K$, and $G_A$ is the group presented by $A$, then there is an isomorphism $\psi : H^2(\Sigma(K), \mathbb{Z}) \to G_A$, with the property that

$$d(\Sigma(K), \beta) = M_A(\psi(\beta))$$

for all $\beta \in H^2(\Sigma(K), \mathbb{Z})$.

For knots with $H(2)$-unknotting number one, we have the following lemma.

**Lemma 2.2** (Montesinos’s trick [9]). If the $H(2)$-unknotting number of a knot $K$ is one, then $\Sigma(K) = \epsilon \cdot S^3_{-p}(C)$ for some knot $C \subset S^3$ and $\epsilon \in \{+1, -1\}$. Here $p = |\det(K)|$ and $S^3_{-p}(C)$ denotes the $-p$-surgery of $S^3$ along the knot $C$.

Proof of Theorem 1.1. If the $H(2)$-unknotting number of $K$ is one, then by Lemma 2.2 $\Sigma(K) = \epsilon \cdot S^3_{-p}(C)$ for some knot $C \subset S^3$ and $\epsilon \in \{+1, -1\}$ and $p = |\det(K)|$. Therefore $\epsilon \cdot \Sigma(K) = \epsilon \cdot S^3_{-p}(C)$ bounds a 4-manifold $X$, which is obtained by attaching a 2-handle to the 4-ball along $C$ with framing $-p$. The intersection form of $X$ is $B = (-p)$. In this case $G_B = \mathbb{Z}/p\mathbb{Z}$, and $X$ is a simply-connected negative-definite 4-manifold.

By (3), there exists a group isomorphism $\varphi : G_B = \mathbb{Z}/p\mathbb{Z} \to H^2(S^3_{-p}(C), \mathbb{Z})$ with

$$d(S^3_{-p}(C), \varphi(i)) = d(\epsilon \cdot \Sigma(K), \varphi(i)) = \epsilon \cdot d(\Sigma(K), \varphi(i)) \geq M_B(i)$$

and

$$\epsilon \cdot d(\Sigma(K), \varphi(i)) \equiv M_B(i) \pmod{2}$$

Theorem 2.1 implies that for the map $\phi = \psi \circ \varphi : \mathbb{Z}/p\mathbb{Z} \to G_A$ (here we automatically identify $H^2(S^3_{-p}(C), \mathbb{Z})$ with $H^2(\Sigma(K), \mathbb{Z})$) we have

$$\epsilon \cdot M_A(\phi(i)) \geq M_B(i)$$

and

$$\epsilon \cdot M_A(\phi(i)) \equiv M_B(i) \pmod{2}$$

for all $i \in \mathbb{Z}/p\mathbb{Z}$. In the following calculation, we abuse $i$ to denote both the element in $\mathbb{Z}/p\mathbb{Z}$ and its representative in the set $\{0, 1, 2, \ldots, p-1\}$. By definition we see that
for any $i \in \mathbb{Z}/p\mathbb{Z}$,

$$M_B(i) = \max \left\{ \frac{u' B^{-1} u + 1}{4} \left|\begin{array}{c} \text{u is odd, [u] = i} \\
\end{array}\right. \right\}$$

$$= \max \left\{ \frac{-u^2 + p}{4p} \left|\begin{array}{c} \text{u is odd, [u] = i} \\
\end{array}\right. \right\}$$

$$= \begin{cases} \frac{-(p - i)^2 + p}{4p} & \text{if } i \text{ is even}, \\
\frac{-(i)^2 + p}{4p} & \text{if } i \text{ is odd.} \\
\end{cases}$$

Writing these two cases in one form we have $M_B(i) = -(1/4)((1/p)((p + (-1)^i p)/2 - i)^2 - 1)$. This completes the proof of Theorem 1.1.

2.3. An example: proof of Corollary 1.2. The pretzel knot $K = P(13, 4, 11)$ is an alternating knot as shown in Fig. 4. A negative-definite Goeritz matrix associated with the mirror image of this diagram is

$$A = \begin{pmatrix} -17 & 4 \\
4 & -15 \end{pmatrix},$$

and the determinant is $\det(A) = \det(K) = 239$. Suppose $G_A$ is the group presented by $A$. In fact, the group $G_A$ is isomorphic to $\mathbb{Z}/239\mathbb{Z}$. In the following calculation, we take the vector $(0, 1)^t$ as a generator of $G_A$. The inverse of the matrix $A$ is

$$A^{-1} = \frac{1}{239} \begin{pmatrix} -15 & -4 \\
-4 & -17 \end{pmatrix}.$$
Then by definition for any $0 \leq r \leq 238$ it holds that

$$M_A((0, r)') = \max\left\{ \frac{(u, v)' A^{-1}(u, v) + 2}{4} \left| (u, v)' \in \text{char}(A), \ [(u, v)'] = (0, r)' \in G_A \right. \right\}$$

$$= \max\left\{ \frac{478 - (15u^2 + 8uv + 17v^2)}{956} \left| u \text{ and } v \text{ are odd, } [(u, v)'] = (0, r)' \in G_A \right. \right\}.$$ 

From this expression, we see that in order to obtain the maximum we only need to focus on those representatives $(u, v)'$ satisfying $|u| \leq 17$ and $|v| \leq 15$.

By calculation, it is easy to see that for any isomorphism $\phi: \mathbb{Z}/239\mathbb{Z} \to \mathbb{Z}/239\mathbb{Z}$ there is

$$I_{\phi_*}(0) = \epsilon \cdot M_A(\phi(0)) + \frac{119}{2} = \epsilon \cdot M_A((0, 0)') + \frac{119}{2} = \frac{\epsilon \cdot (-11) + 119}{2}.$$ 

The vector which realizes the value of $M_A((0, 0)')$ is $(u, v)' = (13, 11)'$ or $(-13, -11)'$.

We assume that $u_2(K) = 1$. Then by Theorem 1.1 the value $I_{\phi_*}(0)$ has to be an even number, and therefore $\epsilon = 1$. Next by calculation we have $I_{\phi_1}(1) = M_A(\phi(1)) - 119/478$. Since 239 is a prime number, any $\phi_j$ = “multiplication by $j$” is an automorphism of $\mathbb{Z}/239\mathbb{Z}$. To guarantee that $I_{\phi_j}(1)$ is an even number, the isomorphism $\phi_j$ has to be either $\phi_{15}$ or $\phi_{224}$. By calculation, we see that

$$I_{\phi_{15,1}}(1) = M_A((0, 15)') - \frac{119}{478} = -4.$$ 

The vector which realizes the value of $M_A((0, 15)')$ is $(u, v)' = (-9, -11)'$. Same calculation tells us that $I_{\phi_{224,1}}(1) = -4$ as well, which is realized by the vector $(u, v)' = (9, 11)'$. Now we see $-4$ is a negative number, which conflicts with the necessary condition stated in Theorem 1.1. Therefore the H(2)-unknotting number of $P(13, 4, 11)$ has to be at least two. On the other hand, the knot $P(13, 4, 11)$ can be changed into the unknot by adding two twisted bands as shown in Fig. 4. Hence the H(2)-unknotting number of $P(13, 4, 11)$ is two. This completes the proof of Corollary 1.2.

### 2.4. Comparisons with other criterions

There have been many criterions and properties which can be used to bound the H(2)-unknotting number of a knot. We want to apply them to the knot $P(13, 4, 11)$ and compare the results with Corollary 1.2.

The first one is Lickorish’s obstruction that we recalled in the beginning. It does not work for the pretzel knot $K = P(13, 4, 11)$ because of the following reason. It is known that the Goeritz matrix $A$ is a presentation matrix of $H_1(\Sigma(K), \mathbb{Z})$, and $A^{-1}$ represents the linking form $\lambda$. It is not hard to see that $H_1(\Sigma(K))$ is cyclic of order 239, and that the generator $g = (0, 1)'$ satisfies $\lambda(g, g) = -17/239$. Then we see

$$\lambda(15g, 15g) = (225 \times (-17))/239 = -3825/239 = -1/239 \text{ over } \mathbb{Q}/\mathbb{Z}.$$ 

Since 239 is a prime number, the vector $g' = (0, 15)'$ can work as a generator of $H_1(\Sigma(K), \mathbb{Z})$. 
There are two invariants of knots which are closely related to H(2)-unknotting number. Given a knot \( K \subseteq S^3 \), the crosscap number of \( K \) \([2]\) is defined as follows:

\[
\gamma(K) = \min \{ \beta_1(F) \mid F \text{ is a non-orientable connected surface in } S^3 \text{ and } \partial F = K \},
\]

where \( \beta_1(F) \) denotes the rank of the first homology group of \( F \). The 4-dimensional crosscap number of \( K \) \([11]\), which we denote \( \gamma^*(K) \) here, is by name defined as follows:

\[
\gamma^*(K) = \min \left\{ \beta_1(F) \mid F \text{ is a non-orientable connected smooth surface in } B^4 \text{ and } \partial F = K \subseteq \partial B^4 = S^3 \right\}.
\]

Their relation with H(2)-unknotting number is as follows.

**Lemma 2.3.** Given a knot \( K \subseteq S^3 \), we have \( \gamma^*(K) \leq u_2(K) \leq \gamma(K) \).

Proof. The knot \( K \) can be reconstructed from the unknot by adding \( u_2(K) \) twisted bands successively. Let \( D \) be a disk bounded by the unknot and \( b_1, b_2, \ldots, b_{u_2(K)} \) be the bands added to the boundary of \( D \). Then \( F := D \cup \bigcup_{i=1}^{u_2(K)} b_i \) is a non-orientable surface in \( B^4 \) with \( \partial F = K \). We have \( \gamma^*(K) \leq \beta_1(F) = u_2(K) \). The second inequality is proved as follows. Suppose \( S \) is a non-orientable surface in \( S^3 \) which realizes the crosscap number of \( K \). Namely we have \( \beta_1(S) = \gamma(K) \) and \( \partial S = K \). Then there are \( \gamma(K) \) disjoint essential arcs in \( S \), say \( \tau_1, \tau_2, \ldots, \tau_{\gamma(K)} \), such that \( S - \tau_i \) has one boundary component for \( i = 1, 2, \ldots, \gamma(K) \) and \( S - \bigcup_{i=1}^{\gamma(K)} \tau_i \) is a disk. If we add twisted bands to \( K \) along \( \tau_i \) for \( i = 1, 2, \ldots, \gamma(K) \), the resulting knot is the unknot. Therefore we have \( u_2(K) \leq \gamma(K) \). \( \square \)

Ichihara and Mizushima \([5]\) calculated the crosscap numbers of pretzel knots. According to their calculation, the crosscap number of \( P(13, 4, 11) \) is two. Gilmer and Livingston \([3]\) studied the 4-dimensional crosscap number of a knot by using Heegaard Floer homology. Their method and our result in this note are both in spirit derived from Theorem 9.6 in \([13]\). The author does not know whether their method can verify that the 4-dimensional crosscap number of \( P(13, 4, 11) \) is 2 or not.

Yasuhara \([18]\), and Kanenobu and Miyazawa \([6]\) introduced some methods for detecting the H(2)-unknotting number of a knot, but simple calculation shows that their methods cannot be applied to the knot \( P(13, 4, 11) \). Taniyama and Yasuhara\([17]\) established the equivalence between H(2)-unknotting number and other two invariants of knots, but there seems no obvious way to apply their relation to the calculation of H(2)-unknotting number.
References


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