Colocal pairs in perfect rings
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Our main aim of the present note is to provide several sufficient conditions for a colocal module $L$ over a left or right perfect ring $A$ to be injective. Also, by developing the previous works [8] and [5], we will extend recent results of Baba [1, Theorems 1 and 2] to left perfect rings and provide simple proofs of them.

Throughout this note, rings are associative rings with identity and modules are unitary modules. For a ring $A$ we denote by $\text{Mod} A$ (resp. $\text{Mod} A^{op}$) the category of left (resp. right) $A$-modules, where $A^{op}$ denotes the opposite ring of $A$. Sometimes, we use the notation $_AL$ (resp. $LA$) to signify that the module $L$ considered is a left (resp. right) $A$-module. For a module $L$, we denote by $\text{soc}(L)$ the socle, by $\text{rad}(L)$ the Jacobson radical, by $E(L)$ an injective envelope and by $\ell(L)$ the composition length of $L$. For a subset $X$ of a right module $LA$ and a subset $M$ of $A$, we set $l_X(M) = \{x \in X | xM = 0\}$ and $r_M(X) = \{a \in M | xa = 0\}$. Also, for a subset $X$ of $A$ and a subset $M$ of a left module $_AL$ we set $l_X(M) = \{a \in X | aM = 0\}$ and $r_M(X) = \{x \in M | Xx = 0\}$. We abbreviate the ascending (resp. descending) chain condition as the ACC (resp. DCC).

Recall that a module $L$ is called colocal if it has simple essential socle. We call a bimodule $_HU_R$ colocal if both $HU$ and $UR$ are colocal. Let $A$ be a semiperfect ring with Jacobson radical $J$. Let $LA$ be a colocal module with $H = \text{End}_A(LA)$ and $f \in A$ a local idempotent with $\text{soc}(LA) \cong fA/fJ$. In case $LA$ has finite Loewy length, we will show that $LA$ is injective if and only if $HLf_{Af}$ is a colocal bimodule and $M = r_{Af}(l_L(M))$ for every submodule $M$ of $Af_{Af}$. Also, in case $A$ is left or right perfect and $\ell(Af/r_{Af}(L)_{fAf}) < \infty$, we will show that the following are equivalent: (1) $LA$ is injective; (2) $HLf_{Af}$ is a colocal bimodule and $r_{Af}(L) = 0$; and (3) $HLf_{Af}$ is a colocal bimodule and $M = r_{Af}(l_L(M))$ for every submodule $M$ of $Af_{Af}$.

Recall that a module $LA$ is called $M$-injective if for any submodule $N$ of $MA$ every $\theta : N_A \to LA$ can be extended to some $\phi : MA \to LA$. Dually, a module $LA$ is called $M$-projective if for any factor module $N$ of $MA$ every $\theta : LA \to N_A$ can be lifted to some $\phi : LA \to MA$. In case $L$ is $L$-injective (resp. $L$-projective), $L$ is called quasi-injective (resp. quasi-projective). Let $A$ be a left perfect ring with Jacobson radical $J$ and $e, f \in A$ local idempotents. Assume $\ell(Af/r_{Af}(eA)_{fAf}) < \infty$. Then we will show that $eA_{AF}$ is quasi-projective with $\text{soc}(eA_{AF}) \cong fA/fJ$ if and only if $E = E(Ae/Je)$ is quasi-projective with $AE/JE \cong Af/Jf$ (cf. [1, Theorem 1]).
We call a pair $(eA, Af)$ of a right ideal $eA$ and a left ideal $Af$ in $A$ a colocal pair if $e, f \in A$ are local idempotents and $eAeAf$ is a colocal bimodule. We will see that $\ell(eAeA/eA(Af)) = \ell(Af/rAf(eA)Af)$ for every colocal pair $(eA, Af)$ in $A$. In case $\ell(eAeA/eA(Af)) = \ell(Af/rAf(eA)Af) < \infty$, a colocal pair $(eA, Af)$ in $A$ is called finite. Let $A$ be a left perfect ring with Jacobson radical $J$ and $e, f_1, f_2, \ldots, f_n \in A$ local idempotents. Put $E = E(eAe/Je)$. Assume $(eA, Af_i)$ is a finite colocal pair in $A$ for all $1 \leq i \leq n$. Then we will show that $\text{soc}(eAe) \cong \bigoplus_{i=1}^n f_i A/f_i J$ if and only if $AE/JE \cong \bigoplus_{i=1}^n Af_i/Jf_i$ (cf. [1, Theorem 2]).

Following Harada [4], we call a module $L_A$ $M$-simple-injective if for any submodule $N$ of $MA$ every $\theta : NA \to L_A$ with $\text{Im} \theta$ simple can be extended to some $\phi : MA \to L_A$. In case $L$ is $L$-simple-injective, $L$ is called simple-quasi-injective. We will show that a left perfect ring $A$ is left artinian if $A$ satisfies the ascending chain condition on annihilator right ideals and $eA_A$ is simple-quasi-injective for every local idempotent $e \in A$.

1. Preliminaries

In this section, we collect several basic results which we need in later sections. We refer to Bass [2] for perfect rings.

**Lemma 1.1.** Let $A$ be a left or right perfect ring and $f \in A$ an idempotent. Assume $\ell(AfAf) < \infty$. Then $A Af$ has finite Loewy length.

**Proof.** Denote by $J$ the Jacobson radical of $A$. Consider first the case of $A$ being left perfect. Since the descending chain $Af \supset Jf \supset \cdots$ terminates, there exists $n \geq 1$ such that $J^n f = J^{n+1} f$. Thus $J^n f = 0$. Assume next that $A$ is right perfect. Then, since the ascending chain $\text{soc}(A Af) \subset \text{soc}^2(A Af) \subset \cdots$ terminates, there exists $n \geq 1$ such that $\text{soc}^n(A Af) = Af$. Thus $J^n f = J^n(\text{soc}^n(A Af)) = 0$. 

**Lemma 1.2.** Let $e \in A$ be an idempotent. Then for a module $L \in \text{Mod} A$ with $r_L(eA) = 0$ the following hold.

1. If $AL$ is simple, so is $eAeL$.
2. $eAeE(AL) \cong E(eAeL)$.
3. The canonical homomorphism $A E(AL) \to A \text{Hom}_{eA}(eA, eE(AL)), x \mapsto (a \mapsto ax)$, is an isomorphism.

**Proof.** (1) See e.g. [5, Lemma 1.1].
(2) See e.g. [5, Lemmas 1.2 and 1.3].
(3) See e.g. [5, Lemma 1.3].

Recall that a module $L_A$ is called $M$-injective if for any submodule $N$ of $MA$ every $\theta : NA \to L_A$ can be extended to some $\phi : MA \to L_A$. Dually, a module $L_A$
is called $M$-projective if for any factor module $N$ of $M_A$ every $\theta : L_A \to N_A$ can be lifted to some $\phi : L_A \to M_A$. In case $L$ is $L$-injective (resp. $L$-projective), $L$ is called quasi-injective (resp. quasi-projective).

**Lemma 1.3** ([6, Theorem 1.1]). Let $L \in \text{Mod} A^{op}$ and put $H = \text{End}_A(E(L_A))$. Then $L_A$ is quasi-injective if and only if $HL = L$. In particular, if $L_A$ is quasi-injective, then we have a surjective ring homomorphism $\rho_L : \text{End}_A(E(L_A)) \to \text{End}_A(L_A)$, $h \mapsto h|_L$.

The equivalence $(1) \leftrightarrow (2)$ of the next lemma is due to Wu and Jans [11, Propositions 2.1, 2.2 and 2.4].

**Lemma 1.4** ([11]). Let $A$ be a left perfect ring. Then for a module $L \in \text{Mod} A$ the following are equivalent.

1. $A_L$ is indecomposable quasi-projective.
2. There exist a local idempotent $f \in A$ and a two-sided ideal $I$ of $A$ such that $A_L \cong A_{tf}$.
3. There exists a local idempotent $f \in A$ such that $A_L \cong A_{tf}/l_{A}(L)f$.

Proof. $(1) \Rightarrow (2)$. By [11, Proposition 2.4] there exists an epimorphism $\pi : A_{tf} \to A_{L}$ with $f \in A$ a local idempotent. Put $K = \text{Ker} \pi$. Then by [11, Proposition 2.2]$K_{tf} = K$ and $A_L \cong A_{tf}/I_{tf}$ with $I = K_{tf} A$ a two-sided ideal of $A$.

$(2) \Rightarrow (1)$. Since $A_{tf}/I_{tf} \cong A_{tf}(I/I)_{tf}$ is projective, $A_{tf}/I_{tf}$ is quasi-projective.

$(2) \Rightarrow (3)$. Since $I_{tf} = l_{A}(A_{tf}/I_{tf})f$, $A_L \cong A_{tf}/l_{A}(L)f$.

$(3) \Rightarrow (2)$. Obvious. \qed

Recall that an object $L$ of an abelian category $A$ in which arbitrary direct products exist is called linearly compact if for any inverse system of epimorphisms $\{\pi_{\lambda} : L \to L_{\lambda}\}_{\lambda \in \Lambda}$ in $A$ the induced morphism $\lim \pi_{\lambda} : L \to \lim L_{\lambda}$ is epic. In case $A = \text{Mod} A$, there is an equivalent definition of linear compactness. Recall that, for a family of submodules $\{L_{\lambda}\}_{\lambda \in \Lambda}$ in a module $A_L$, a system of congruences $\{x \equiv x_{\lambda} \text{mod} L_{\lambda}\}_{\lambda \in \Lambda}$ is said to be finitely solvable if for any nonempty finite subset $F$ of $\Lambda$ there exists $x_{F} \in L$ such that $x_{F} \equiv x_{\lambda} \text{mod} L_{\lambda}$ for all $\lambda \in F$, and to be solvable if there exists $x_{0} \in L$ such that $x_{0} \equiv x_{\lambda} \text{mod} L_{\lambda}$ for all $\lambda \in \Lambda$.

For the benefit of the reader, we include a proof of the following.

**Proposition 1.5.** For a module $L \in \text{Mod} A$ the following are equivalent.

1. $A_L$ is linearly compact.
2. For any family of submodules $\{L_{\lambda}\}_{\lambda \in \Lambda}$ in $A_L$, every finitely solvable system of congruences $\{x \equiv x_{\lambda} \text{mod} L_{\lambda}\}_{\lambda \in \Lambda}$ is solvable.
Proof. (1) ⇒ (2). Let \( \{L_\lambda\}_{\lambda \in \Lambda} \) be a family of submodules in \( L \) and \( \{x \equiv x_\lambda \mod L_\lambda\}_{\lambda \in \Lambda} \) a finitely solvable system of congruences. Denote by \( \phi_\lambda : L \rightarrow L/L_\lambda \) the canonical epimorphism for each \( \lambda \in \Lambda \) and set \( \hat{x} : L \rightarrow \prod_{\lambda \in \Lambda} L/L_\lambda, x \mapsto (\phi_\lambda(x)) \). Put \( \hat{x} = (\phi_\lambda(x_\lambda)) \in \prod_{\lambda \in \Lambda} L/L_\lambda \). We claim that \( \hat{x} \in \text{Im} \phi_\lambda \). Let \( F \) be the directed set of nonempty finite subsets of \( \Lambda \). For each \( F \in \mathcal{F} \), denote by \( p_F : \prod_{\lambda \in \Lambda} L/L_\lambda \rightarrow \prod_{\lambda \in F} L/L_\lambda \) the projection and put \( \hat{x}_F = p_F(\hat{x}) \in \prod_{\lambda \in F} L/L_\lambda \) and \( X_F = (p_F \circ \phi)^{-1}(A\hat{x}_F) \). Note that for any \( F \in \mathcal{F} \), since \( \{x \equiv x_\lambda \mod L_\lambda\}_{\lambda \in \Lambda} \) is finitely solvable, \( p_F \circ \phi : L \rightarrow \prod_{\lambda \in F} L/L_\lambda \) induces an epimorphism \( \phi_F : X_F \rightarrow A\hat{x}_F \). For each \( F \in \mathcal{F} \), take a push-out of \( \phi_F : X_F \rightarrow A\hat{x}_F \) along with the inclusion \( X_F \rightarrow L \):

\[
\begin{array}{ccc}
X_F & \longrightarrow & L \\
\varphi_F \downarrow & & \downarrow \pi_F \\
A\hat{x}_F & \longrightarrow & Y_F.
\end{array}
\]

Then we get an inverse system of epimorphisms \( \{\varphi_F : L \rightarrow Y_F\}_{F \in \mathcal{F}} \). Also, since \( \lim \) is left exact, we get a pull-back square

\[
\begin{array}{ccc}
\lim X_F & \longrightarrow & L \\
\lim \varphi_F \downarrow & & \downarrow \lim \pi_F \\
\lim A\hat{x}_F & \longrightarrow & \lim Y_F.
\end{array}
\]

Since \( L \) is linearly compact, \( \lim \pi_F \) is epic, so is \( \lim \varphi_F \). Note that \( \lim X_F \rightarrow \bigcap_{F \in \mathcal{F}} X_F \). Also, \( \lim p_F : \prod_{\lambda \in \Lambda} L/L_\lambda \rightarrow \lim \prod_{\lambda \in F} L/L_\lambda \) is an isomorphism and hence induces an isomorphism \( A\hat{x} \rightarrow \lim A\hat{x}_F \). It follows that \( \phi(\bigcap_{F \in \mathcal{F}} X_F) = A\hat{x} \). Thus \( \hat{x} \in \text{Im} \phi \).

(2) ⇒ (1). Let \( \{\pi_\lambda : L \rightarrow L_\lambda\}_{\lambda \in \Lambda} \) be an inverse system of epimorphisms in \( \text{Mod} A \). We may consider \( \lim L_\lambda \) as a submodule of \( \prod_{\lambda \in \Lambda} L_\lambda \). Let \( (x_\lambda) \in \lim L_\lambda \) and for each \( \lambda \in \Lambda \) choose \( y_\lambda \in L \) with \( \pi_\lambda(y_\lambda) = x_\lambda \). Then, since for any nonempty finite subset \( F \) of \( \Lambda \) there exists \( \lambda_0 \in \Lambda \) such that \( \lambda_0 \geq \lambda \) for all \( \lambda \in F \), the system of congruences \( \{x \equiv y_\lambda \mod \ker \pi_\lambda\}_{\lambda \in \Lambda} \) is finitely solvable and thus solvable. Hence \( \lim \pi_\lambda : L \rightarrow \lim L_\lambda \) is an epimorphism. \( \square \)

Let \( HU_R \) be a bimodule and \( K \in \text{Mod} R^{op} \). For a pair of a subset \( X \) of \( (K_R)^* \) and a subset \( M \) of \( K_R \), we set \( r_M(X) = \{a \in M|h(a) = 0 \text{ for all } h \in X\} \) and \( l_X(M) = \{h \in X|h(a) = 0 \text{ for all } a \in M\} \), where \( (\ )^* = \text{Hom}_R(\cdot, HU_R) \).

The next lemma is due essentially to [7, Lemma 4].

**Lemma 1.6.** Let \( HU_R \) be a bimodule and \( K \in \text{Mod} R^{op} \) a module such that \( U_R \) is \( K \)-injective. Assume \( X = l_{K^*}(r_K(X)) \) for every submodule \( X \) of \( (K_R)^* \). Then \( (K_R)^* \) is linearly compact.
Proof. Let \( \{\pi_\lambda : K^* \to X_\lambda\}_{\lambda \in \Lambda} \) be an inverse system of epimorphisms in Mod \( H \). For \( \lambda \in \Lambda \), put \( Y_\lambda = \text{Ker} \pi_\lambda \) and \( M_\lambda = r_K(Y_\lambda) \), and let \( j_\lambda : M_\lambda \to K \) be the inclusion. Then for each \( \lambda \in \Lambda \), since \( \text{Ker} j_\lambda^* \cong l_{K^*}(M_\lambda) = Y_\lambda \), and since \( j_\lambda^* : K^* \to M_\lambda^* \) is epic, there exists an isomorphism \( \phi_\lambda : M_\lambda^* \to X_\lambda \) with \( \pi_\lambda = \phi_\lambda \circ j_\lambda^* \). Since \( \lim j_\lambda \) is monic, \( \lim j_\lambda^* \) is epic. Also, \( \lim \pi_\lambda \) is an isomorphism. Thus \( \lim \pi_\lambda = (\lim \phi_\lambda) \circ (\lim j_\lambda^*) \) is epic.

Corollary 1.7. Let \( A \) be a left or right perfect ring. Assume \( A_A \) is injective and \( I = l_A(r_A(I)) \) for every left ideal \( I \) of \( A \). Then \( A \) is quasi-Frobenius.

Proof. It follows by Lemma 1.6 that \( A_A \) is linearly compact. Thus by [10, Propositions 2.9 and 2.12] \( A \) is left noetherian.

2. Bilinear maps into colocal bimodules

In this section, as further preliminaries, we modify results of [8, Section 1]. For a left \( H \)-module \( H_L \), a right \( R \)-module \( H_R \) and an \( \oplus_R \)-bimodule \( H_{UR} \), we call a map \( \varphi : H_L \times K_R \to H_{UR} \) \( \oplus_R \)-bilinear if \( K_R \to H_{UR}, a \mapsto \varphi(x, a) \), is \( \oplus_R \)-linear for every \( x \in L \) and \( H_L \to H \text{Hom}_R(K_R, H_{UR}), x \mapsto (a \mapsto \varphi(x, a)) \), is \( H \)-linear.

Throughout this section, \( \varphi : H_L \times K_R \to H_{UR} \) is a fixed \( \oplus_R \)-bilinear map. For a pair of a subset \( X \) of \( L \) and a subset \( M \) of \( R \), we set \( V_M(X) = \{a \in M | \varphi(x, a) = 0 \text{ for all } x \in X\} \) and \( Z_X(M) = \{x \in X | \varphi(x, a) = 0 \text{ for all } a \in M\} \). We denote by \( \mathcal{A}_l(L, K) \) the lattice of submodules \( X \) of \( H_L \) with \( X = l_L(r_K(X)) \) and by \( \mathcal{A}_r(L, K) \) the lattice of submodules \( M \) of \( K_R \) with \( M = r_K(l_L(M)) \).

REMARKS (see e.g. [3, Part 1] for details). (1) Let \( X \) be a subset of \( L \). Then \( \varphi(X, r_K(X)) = 0 \) implies \( X \subseteq l_L(r_K(X)) \) and thus \( r_K(l_L(r_K(X))) \subseteq r_K(X) \). Also, \( \varphi(l_L(r_K(X)), r_K(X)) = 0 \) implies \( r_K(X) \subseteq r_K(l_L(r_K(X))) \). Thus \( r_K(X) = r_K(l_L(r_K(X))) \) and \( r_K(X) \in \mathcal{A}_l(L, K) \).

(2) Let \( X \) be a subset of \( L \). For any \( Y \in \mathcal{A}_l(L, K) \) with \( X \subseteq Y \), \( l_L(r_K(Y)) \subseteq l_L(r_K(X)) = Y \). Thus \( l_L(r_K(X)) \) is the smallest module in \( \mathcal{A}_l(L, K) \) containing \( X \).

(3) Let \( \{X_\lambda\}_{\lambda \in \Lambda} \) be a family of submodules of \( H_L \). For any \( \mu \in \Lambda \), since \( \bigcap_{\lambda \in \Lambda} X_\lambda \subseteq X_\mu \subseteq \sum_{\lambda \in \Lambda} X_\lambda \), \( r_K(\sum_{\lambda \in \Lambda} X_\lambda) \subseteq r_K(X_\mu) \subseteq r_K(\bigcap_{\lambda \in \Lambda} X_\lambda) \). Thus \( r_K(\sum_{\lambda \in \Lambda} X_\lambda) \subseteq \bigcap_{\lambda \in \Lambda} r_K(X_\lambda) \) and \( \sum_{\lambda \in \Lambda} r_K(X_\lambda) \subseteq r_K(\bigcap_{\lambda \in \Lambda} X_\lambda) \). Let \( a \in \bigcap_{\lambda \in \Lambda} r_K(X_\lambda) \). Since \( \varphi(X_\lambda, a) = 0 \text{ for all } \lambda \in \Lambda \), and since \( H_L \to H_U, x \mapsto \varphi(x, a) \), is \( H \)-linear, \( \varphi(\sum_{\lambda \in \Lambda} X_\lambda, a) = 0 \) and \( a \in r_K(\sum_{\lambda \in \Lambda} X_\lambda) \). Thus \( r_K(\sum_{\lambda \in \Lambda} X_\lambda) = \bigcap_{\lambda \in \Lambda} r_K(X_\lambda) \).

(4) Let \( \{X_\lambda\}_{\lambda \in \Lambda} \) be a family of submodules of \( H_L \) with the \( X_\lambda \in \mathcal{A}_l(L, K) \). Then by (3) \( \bigcap_{\lambda \in \Lambda} X_\lambda = \bigcap_{\lambda \in \Lambda} l_L(r_K(X_\lambda)) = l_L(\sum_{\lambda \in \Lambda} r_K(X_\lambda)) \). Thus \( r_K(\bigcap_{\lambda \in \Lambda} X_\lambda) = r_K(l_L(\sum_{\lambda \in \Lambda} r_K(X_\lambda))) \) and by (2) \( r_K(\bigcap_{\lambda \in \Lambda} X_\lambda) \) is the smallest module in \( \mathcal{A}_r(L, K) \) containing \( \sum_{\lambda \in \Lambda} r_K(X_\lambda) \), so that \( r_K(\bigcap_{\lambda \in \Lambda} X_\lambda) = \sum_{\lambda \in \Lambda} r_K(X_\lambda) \) whenever \( \sum_{\lambda \in \Lambda} r_K(X_\lambda) \in \mathcal{A}_r(L, K) \).
(5) We have an anti-isomorphism of lattices \( \mathcal{A}_i(L, K) \to \mathcal{A}_r(L, K), X \mapsto r_K(X) \).
In particular, \( \mathcal{A}_i(L, K) \) satisfies the ACC (resp. DCC) if and only if \( \mathcal{A}_r(L, K) \) satisfies
the DCC (resp. ACC).

Recall that a module is called colocal if it has simple essential socle. We call a
bimodule \( HU_R \) colocal if both \( HU \) and \( U_R \) are colocal modules.

**Lemma 2.1.** Let \( HU_R \) be a colocal bimodule. Then \( \text{soc}(HU) = \text{soc}(UR) \).

**Proof.** Since \( \text{soc}(HU) \) is a subbimodule of \( HU_R, \text{soc}(UR) \subseteq \text{soc}(HU) \).
Similarly, \( \text{soc}(HU) \subseteq \text{soc}(UR) \). Thus \( \text{soc}(HU) = \text{soc}(UR) \).

Throughout the rest of this section, \( HU_R \) is assumed to be a colocal bimodule with\( HSR = \text{soc}(HU) = \text{soc}(UR) \), and \( (\cdot)^* \) denotes both
the \( U \)-dual functors.

**Lemma 2.2.** The following hold.
(1) The canonical ring homomorphisms \( H \to \text{End}_R(S_R) \) and \( R \to \text{End}_H(HS)^{\text{op}} \)
are surjective.
(2) \( (HS)^* \cong SR \) and \( (SR)^* \cong HS \).

**Proof.** (1) Let \( 0 \neq u \in S \). Then \( S = Hu = uR. \) For any \( h \in \text{End}_R(S_R), \)
\( h(u) = au \) for some \( a \in H \) and \( h(ub) = h(u)b = (au)b = a(ub) \) for all \( b \in R. \)
Thus the canonical ring homomorphism \( H \to \text{End}_R(S_R) \) is surjective. Similarly, the
canonical ring homomorphism \( R \to \text{End}_H(HS)^{\text{op}} \) is surjective.

(2) Let \( \pi : R_R \to S_R \) be an epimorphism. We have a monomorphism \( \mu : (S_R)^* \to HU \) such that \( \mu(h) = (\pi^*(h))(1) \) for \( h \in (S_R)^* \). Put \( u = \pi(1) \). Then \( \mu(h) = h(u) \in S \)\nfor all \( h \in (S_R)^* \) and \( \text{Im}\mu = HS \), so that \( (S_R)^* \cong HS \). Similarly, \( (HS)^* \cong SR \).

**Lemma 2.3.** Let \( N \subseteq M \) be submodules of \( K_R \) with \( N = r_K(l_L(N)) \) and
\( M/N \) simple. Then the following hold.
(1) \( M/N \cong S_R \) and \( l_L(N)/l_L(M) \cong (M/N)^* \cong HS \).
(2) \( M = r_K(l_L(M)) \).

**Proof.** (1) Since \( M \neq N = r_K(l_L(N)) \), \( l_L(M) \subseteq l_L(N) \) with \( l_L(N)/l_L(M) \neq 0. \) Let \( j : N_R \to M_R \) be the inclusion. Then we have the following commutative
diagram with exact rows:

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & l_L(M) & \longrightarrow & L & \longrightarrow & M^* \\
& & \downarrow & & \| & & \downarrow j^* \\
0 & \longrightarrow & l_L(N) & \longrightarrow & L & \longrightarrow & N^*.
\end{array}
\]
Thus $0 \neq l_L(N)/l_L(M)$ embeds in $\text{Ker} j^* \cong (M/N)^*$. Hence $(M/N)^* \neq 0$, so that $M/N \cong S_R$ and by Lemma 2.2(2) $(M/N)^* \cong H_S$.

(2) Since $l_L(M) \subset l_L(N)$ with $l_L(N)/l_L(M)$ simple, one can apply the part (1) to conclude that $r_K(l_L(M))/r_K(l_L(N))$ is simple. Thus, since $r_K(l_L(N)) = N \subset M \subset r_K(l_L(M))$ with both $M/N$ and $r_K(l_L(M))/r_K(l_L(N))$ simple, it follows that $M = r_K(l_L(M))$. 

**Lemma 2.4.** Let $M$ be a submodule of $K_R$ with $r_K(L) \subset M$ and $\ell(M/r_K(L)_R) < \infty$. Then the following hold.

1. Every composition factor of $M/r_K(L)_R$ is isomorphic to $S_R$.
2. $M = r_K(l_L(M))$.

**Proof.** Since $r_K(L) = r_K(l_L(r_K(L)))$, Lemma 2.3 enables us to make use of induction on $\ell(M/r_K(L)_R)$. 

**Lemma 2.5** ([8, Lemma 1.3]). $\ell(H_L/l_L(K)) = \ell(K/r_K(L)_R)$. 

**Proof.** By symmetry we may assume $\ell(H_L/l_L(K)) < \infty$. Let $l_L(K) = L_0 \subset L_1 \subset \cdots \subset L_n = L$ be a chain of submodules of $H_L$ with the $L_{i+1}/L_i$ simple. Then by Lemma 2.3 we get a chain of submodules $r_K(l_L(L)) = r_K(L_n) \subset \cdots \subset r_K(L_1) \subset r_K(L_0) = K$ in $K_R$ with the $r_K(L_i)/r_K(L_{i+1})$ simple. 

**Lemma 2.6.** Assume $R$ is left perfect. Then the following are equivalent.

1. $\ell(K/r_K(L)_R) < \infty$.
2. $A_T(L, K)$ satisfies both the ACC and the DCC.
3. $A_T(L, K)$ satisfies the ACC.

**Proof.** (1) $\Rightarrow$ (2) $\Rightarrow$ (3). Obvious.

(3) $\Rightarrow$ (1). It follows by Lemma 2.4 that there exists a maximal element $K_0$ in the set of submodules $M$ of $K_R$ with $r_K(L) \subset M$ and $\ell(M/r_K(L)_R) < \infty$. We claim $K_0 = K$. Otherwise, there exists a submodule $M$ of $K_R$ with $K_0 \subset M$ and $M/K_0$ simple, a contradiction. 

3. Simple-injective colocal modules

Throughout the rest of this note, $A$ stands for a ring with Jacobson radical $J$. For any pair of a right module $L_A$ and a left ideal $K$ of $A$, we have a canonical bilinear map $H_L \times K_R \rightarrow H_LK_R$, $(x, a) \mapsto xa$, where $H = \text{End}_A(L_A)$ and $R = \text{End}_A(AK)^{op}$, so that, in case $H_LK_R$ is a colocal bimodule, we can apply results of the preceding section.
**Lemma 3.1.** Let $L \in \text{Mod } A^{op}$ be a colocal module and $f \in A$ a local idempotent with $\text{soc}(L_A) \cong fA/fJ$. Then the following hold.

1. $l_L(Af) = 0$.
2. $l_L(If) = l_L(I)$ for every right ideal $I$ of $A$.
3. $L_{fAf}$ is colocal with $\text{soc}(L_{fAf}) = \text{soc}(L_A)f$.

**Proof.** (1) For any $0 \neq x \in L$, since $\text{soc}(L_A) \subset xA$, $0 \neq \text{soc}(L_A)f \subset xAf$ and thus $x \notin l_L(Af)$.

(2) We have $l_L(I) \subset l_L(If)$. For any $x \in l_L(If)$, since $xAf = xIf = 0$, by the part (1) $xI \subset l_L(Af) = 0$ and $x \in l_L(I)$. Thus $l_L(If) \subset l_L(I)$.

(3) Let $0 \neq x \in \text{soc}(L_A)f$. For any $0 \neq y \in Lf$, since $xA \subset yA$, $xfAf = xAf \subset yAf = yfAf$. Thus $L_{fAf}$ is colocal and $\text{soc}(L_{fAf}) = \text{soc}(L_A)f$.

**Lemma 3.2.** Let $L \in \text{Mod } A^{op}$ and $f \in A$ a local idempotent. Then the following are equivalent.

1. $L_A$ is colocal with $\text{soc}(L_A) \cong fA/fJ$.
2. $L_{fAf}$ is colocal and $l_L(Af) = 0$.

**Proof.** (1) $\Rightarrow$ (2). By (3) and (1) of Lemma 3.1.

(2) $\Rightarrow$ (1). Since by Lemma 1.2(2) $E(L_A)f_{fAf} \cong E(L_{fAf}) \cong E(fAf/fJf_{fAf}) \cong E(fA/fJ_A)f_{fAf}$, by Lemma 1.2(3) $E(L_A) \cong \text{Hom}_{fAF}(Af,E(L_A)f_A) \cong \text{Hom}_{fAF}(Af,E(fA/fJ_A)f_A) \cong E(fA/fJ_A)$. Thus $L_A$ is colocal with $\text{soc}(L_A) \cong fA/fJ$.

**Corollary 3.3.** Let $e, f \in A$ be local idempotents. Then the following are equivalent.

1. $eA/l_{eA}(Af)_A$ is colocal with $\text{soc}(eA/l_{eA}(Af)_A) \cong fA/fJ$.
2. $eAf_{fAf}$ is colocal.

**Proof.** Put $L = eA/l_{eA}(Af)_A$. Then $l_L(Af) = 0$ and, since $l_{eA}(Af)f = 0$, $L_{fAf} \cong eAf_{fAf}$. Thus Lemma 3.2 applies.

Following Harada [4], we call a module $L_A$ $M$-simple-injective if for any submodule $N$ of $M_A$ every $\theta : N_A \rightarrow L_A$ with $\text{Im} \theta$ simple can be extended to some $\phi : M_A \rightarrow L_A$. In case $L$ is $L$-simple-injective, $L$ is called simple-quasi-injective.

**Lemma 3.4.** Let $L \in \text{Mod } A^{op}$ be a colocal module and put $H = \text{End}_A(L_A)$. Let $f \in A$ be a local idempotent with $\text{soc}(L_A) \cong fA/fJ$. Then the following hold.

1. If $L_A$ is $A$-simple-injective, then $M \cong r_{Af}(l_L(M))$ for every submodule $M$ of $A_{fAf}$.
2. If $H_{fAf}$ is a colocal bimodule and $M \cong r_{Af}(l_L(M))$ for every submodule $M$ of $A_{fAf}$, then $L_A$ is $A$-simple-injective.
Proof. (1) Let $M$ be a submodule of $AfAf$ and put $N = rAf(l_L(M))$. We claim $M = N$. Suppose otherwise. Note first that $l_L(N) = l_L(M)$. Since $(NA/MA)f \cong N/M \neq 0$, there exist right ideals $K$, $I$ of $A$ such that $MA \subset K \subset I \subset NA$ and $I/K \cong fAf/J \cong \text{soc}(L_A)$. Then we have $\theta : I_A \rightarrow L_A$ with $\text{Im} \theta = \text{soc}(L_A)$ and $\text{Ker} \theta = K$. Let $\mu : I_A \rightarrow A_A$ be the inclusion. There exists $\phi : A_A \rightarrow L_A$ with $\phi \circ \mu = \theta$. Then $\phi(I) = \phi(I) = \theta(I) \neq 0$ and $\phi(1)K = \phi(K) = \theta(K) = 0$. Thus $\phi(1) \in l_L(K)$ and $\phi(1) \notin l_L(I)$. Since $l_L(N) = l_L(NA) \subset l_L(I) \subset l_L(K) \subset l_L(MA) = l_L(M)$, $l_L(K) \neq l_L(I)$ implies $l_L(M) \neq l_L(N)$, a contradiction.

(2) Let $I$ be a nonzero right ideal of $A$ and $\mu : I_A \rightarrow A_A$ the inclusion. Let $\theta : I_A \rightarrow L^A$ with $\text{Im} \theta = \text{soc}(L^A)$ and put $\Lambda = \text{Ker} \theta$. Since by Lemma 1.2(1) $lf/KfAf \cong (I/K)fAf$ is simple, by Lemma 2.3(1) so is $Hl_L(Kf)/l_L(If)$. Let $a \in If$ with $a \notin Kf$. Then, since $l_L(Kf)a \neq 0$ and $l_L(If)a = 0$, $Hl_L(Kf)a$ is simple. Thus by Lemmas 2.1 and 3.1(3) $l_L(Kf)a = \text{soc}(L_ffAf) = \text{soc}(L^A)$, so that $\theta(a) = \theta(af) = \theta(a)f = xa$ for some $x \in l_L(Kf)$. Define $\phi : A_A \rightarrow L_A$ by $1 \mapsto x$. Then, since by Lemma 3.1(2) $x \in l_L(Kf) = l_L(K)$, and since $I = K + aA$, we have $\phi \circ \mu = \theta$. $
$
Lemma 3.5. Let $L \in \text{Mod} A^{op}$ be a colocal module and put $H = \text{End}_A(L_A)$. Let $f \in A$ be a local idempotent with $\text{soc}(L_A) \cong fAf/Jf$. Then the following hold.

(1) If $L_A$ is simple-quasi-injective, then $HLfAf$ is a colocal bimodule and $l_L(Af) = 0$.

(2) If $L_A$ is $A$-simple-injective, then $rAf(L) = 0$ and $r_A(L/LJ_A) \subset l_A(\text{soc}(A Af))$.

Proof. (1) By Lemma 3.2 $LfAf$ is colocal and $l_L(Af) = 0$. Let $0 \neq x \in \text{soc}(L_A)f$. We claim that $x \in Hf$ for all $0 \neq y \in Lf$. Note that $rAf(x) = fJ$. Let $0 \neq y \in Lf$. Then $rAf(y) \in fJ = rAf(x)$ and we have $\theta : yA_A \rightarrow xA_A = \text{soc}(L_A)$, $ya \mapsto xa$. Let $\mu : \text{soc}(L_A) \rightarrow L_A$ and $\nu : yA_A \rightarrow L_A$ be inclusions. There exists $h \in H$ with $h \circ \nu = \mu \circ \theta$, so that $x = h(y) \in Hf$. Thus $HLf$ is colocal.

(2) By Lemma 3.4(1) $rAf(L) = rAf(l_L(0)) = 0$. Next, let $a \in r_A(L/LJ)$. Since $La \subset LJ$, $La(\text{soc}(A Af)) \subset LJ(\text{soc}(A Af)) = 0$. Thus $a(\text{soc}(A Af)) \subset rAf(L) = 0$ and $a \in l_A(\text{soc}(A Af))$.

Lemma 3.6 ([5, Lemma 3.3]). Let $L \in \text{Mod} A^{op}$ be a simple-quasi-injective module with $\text{soc}(L_A) \neq 0$. Assume $\text{End}_A(L_A)$ is a local ring. Then $\text{soc}(L_A)$ is simple.

Proof. Let $S$ be a simple submodule of $\text{soc}(L_A)$. Suppose to the contrary that $S \neq \text{soc}(L_A)$. Let $\pi : \text{soc}(L_A) \rightarrow S_A$ be a projection and $\mu : \text{soc}(L_A) \rightarrow L_A$, $\nu : S_A \rightarrow L_A$ inclusions. There exists $\phi : L_A \rightarrow L_A$ with $\phi \circ \mu = \nu \circ \pi$. Since $\pi$ is not monic, $\phi$ is not an isomorphism. Thus $\phi \in \text{rad} \text{End}_A(L_A)$ and $(\text{id}_L - \phi)$ is a unit in $\text{End}_A(L_A)$, so that for $0 \neq x \in S$, since $\phi(x) = \pi(x) = x$, $(\text{id}_L - \phi)(x) = 0$ and $x = 0$, a contradiction.
4. Injectivity of colocal modules

In this section, by extending the previous results [8, Theorems 2.7, 2.8 and Proposition 2.9], we provide several sufficient conditions for a colocal module over a left or right perfect ring $A$ to be injective.

**Lemma 4.1** ([5, Lemma 3.4]). Let $A$ be a semiperfect ring and $L \in \text{Mod } A^{\text{op}}$ an $A$-simple-injective colocal module of finite Loewy length. Then $L_A$ is injective.

Proof. Let $I$ be a right ideal of $A$ and $\mu : I_A \rightarrow A_A$ the inclusion. Let $\theta : I_A \rightarrow L_A$. We make use of induction on the Loewy length of $\theta(I)$ to show the existence of $\phi : A_A \rightarrow L_A$ with $\theta = \phi \circ \mu$. Let $n = \min\{k \geq 0 | \theta(I)J^k = 0\}$. We may assume $n > 0$. Since $\text{soc}(L_A)$ is simple, $\text{soc}(L_A) = \theta(I)J^{n-1} = \theta(IJ^{n-1})$. Let $\mu_1$ and $\theta_1$ denote the restrictions of $\mu$ and $\theta$ to $IJ^{n-1}$, respectively. Then $\text{Im} \theta_1 = \text{soc}(L_A)$ and there exists $\phi_1 : A_A \rightarrow L_A$ with $\phi_1 \circ \mu_1 = \theta_1$. Since $(\theta - \phi_1 \circ \mu)(IJ^{n-1}) = 0$, by induction hypothesis there exists $\phi_2 : A_A \rightarrow L_A$ with $\phi_2 \circ \mu = \theta - \phi_1 \circ \mu$. Thus $\theta = (\phi_1 + \phi_2) \circ \mu$. 

**Theorem 4.2.** Let $A$ be a semiperfect ring. Let $L \in \text{Mod } A^{\text{op}}$ be a colocal module of finite Loewy length and put $H = \text{End}_A(L_A)$. Let $f \in A$ be a local idempotent with $\text{soc}(L_A) \cong fA/fJ$. Then the following are equivalent.

1. $L_A$ is injective.
2. $HfAf$ is a colocal bimodule and $M = \tau_{fAf}(l_L(M))$ for every submodule $M$ of $AfAf$.

Proof. (1) $\Rightarrow$ (2). By Lemmas 3.5(1) and 3.4(1).

(2) $\Rightarrow$ (1). By Lemmas 3.4(2) and 4.1.

**Corollary 4.3.** Let $A$ be a semiperfect ring. Let $L \in \text{Mod } A^{\text{op}}$ be a colocal module of finite Loewy length and put $H = \text{End}_A(L_A)$. Let $f \in A$ be a local idempotent with $\text{soc}(L_A) \cong fA/fJ$. Assume $HfAf$ is a colocal bimodule and $M = \tau_{fAf}(l_L(M))$ for every submodule $M$ of $AfAf$ with $\tau_{fAf}(L) \subseteq M$. Then $L_A$ is quasi-injective.

Proof. Put $I = \tau_{fAf}(L)$. Then by Theorem 4.2 $L_{A/I}$ is injective, so that $L_A$ is quasi-injective.

**Theorem 4.4.** Let $A$ be a left or right perfect ring. Let $L \in \text{Mod } A^{\text{op}}$ be a colocal module and put $H = \text{End}_A(L_A)$. Let $f \in A$ be a local idempotent with $\text{soc}(L_A) \cong fA/fJ$. Assume $\ell(Af/\tau_{fAf}(L)Af) < \infty$. Then the following are equivalent.

1. $L_A$ is injective.
2. $HfAf$ is a colocal bimodule and $\tau_{fAf}(L) = 0$. 


(3) $H L f_{|A^f}$ is a colocal bimodule and $M = r_{A^f}(l_L(M))$ for every submodule $M$ of $A^f_{|A^f}$.

Proof. (1) $\Rightarrow$ (2). By Lemma 3.5.
(2) $\Rightarrow$ (3). By Lemma 2.4.
(3) $\Rightarrow$ (1). By Lemma 3.4(2) $L_A$ is $A$-simple-injective. Note that $r_{A^f}(L) = r_{A^f}(l_L(0)) = 0$. Thus $\ell(A f_{|A^f}) < \infty$ and by Lemma 1.1 $J^n f = 0$ for some $n \geq 1$, so that $LJ^n A^f = LJ^n f = 0$ and by Lemma 3.1(1) $LJ^n \subseteq l_L(A^f) = 0$. Hence by Lemma 4.1 $L_A$ is injective.

Corollary 4.5. Let $A$ be a left or right perfect ring. Let $L \in \text{Mod} A^{op}$ be a colocal module and put $H = \text{End}_A(L_A)$. Let $f \in A$ be a local idempotent with $\text{soc}(L_A) = fA/fJ$. Assume $H f f_{|A^f}$ is a colocal bimodule and $\ell(A f/r_{A^f}(L)f_{|A^f}) < \infty$. Then $L_A$ is quasi-injective.

Proof. Put $I = r_A(L)$. Then $r_{A^f/I^f}(L) = 0$ and by Theorem 4.4 $L_{A/I}$ is injective, so that $L_A$ is quasi-injective.

Proposition 4.6. Let $A$ be a left or right perfect ring. Let $L \in \text{Mod} A^{op}$ be a colocal module and put $H = \text{End}_A(L_A)$. Let $f \in A$ be a local idempotent with $\text{soc}(L_A) = fA/fJ$. Then the following are equivalent.
(1) $L_A$ is injective and $X = l_L(r_{A^f}(X))$ for every submodule $X$ of $H L$.
(2) $H L f_{|A^f}$ is a colocal bimodule, $r_{A^f}(L) = 0$ and $\ell(A f_{|A^f}) < \infty$.

Proof. (1) $\Rightarrow$ (2). By Lemma 3.5(1) $H L f_{|A^f}$ is a colocal bimodule, and by Lemma 3.5(2) $r_{A^f}(L) = 0$. It remains to show $\ell(A f_{|A^f}) < \infty$. Put $K_n = A^f(f^n f)$ for $n \geq 0$. We claim $\ell(K_n/K_{n+1} f_{|A^f}) < \infty$ for all $n \geq 0$. Let $n \geq 0$. Note that by Lemma 3.4(1) the lattice of submodules of $A^f_{|A^f}$ is anti-isomorphic to the lattice of submodules of $H L$. Thus $\ell(K_n/K_{n+1} f_{|A^f}) = \ell(H l_L(K_{n+1})/l_L(K_n))$. Also, since $\text{rad}(K_n/K_{n+1} f_{|A^f}) = 0$, $H l_L(K_{n+1})/l_L(K_n)$ is semisimple. For any submodule $X$ of $H L$, since $r_{A^f}(X) = r_A(X)f$, by Lemma 3.1(2) $X = l_L(r_{A^f}(X)) = l_L(r_A(X)f) = l_L(r_A(X))$. Thus by Lemma 1.6 $H L \cong \text{Hom}_A(A_A, H L_A)$ is linearly compact, so is $H l_L(K_{n+1})/l_L(K_n)$ by [10, Proposition 2.2]. Hence by [10, Lemma 2.3] $\ell(K_n/K_{n+1} f_{|A^f}) = \ell(H l_L(K_{n+1})/l_L(K_n)) < \infty$. Since $\ell(f J^2 f/((f J f)^2 f_{|A^f})) < \ell(K_1/K_{2f_{|A^f}}) < \infty$, by [9, Lemma 11] $f A^f$ is right artinian. Then $\ell(K_0/K_{1f_{|A^f}}) < \infty$ implies $\ell(A f_{|A^f}) < \infty$.

(2) $\Rightarrow$ (1). By Theorem 4.4 $L_A$ is injective. Since by Lemma 3.1(1) $l_L(A^f) = 0$, by Lemma 2.5 $\ell(H L) = \ell(A f_{|A^f}) < \infty$ and thus by Lemma 2.4 $X = l_L(r_{A^f}(X))$ for every submodule $X$ of $H L$.
5. Colocal pairs

We call a pair \((eA, Af)\) of a right ideal \(eA\) and a left ideal \(Af\) in \(A\) a colocal pair if \(e, f \in A\) are local idempotents and \(eAeAfA\) is a colocal bimodule. Note that by Lemma 2.5 \(\ell(eAeA/eA(Af)) = \ell(Af/rAf(eA)fA)\) for every colocal pair \((eA, Af)\) in \(A\). In case \(eAeA/eA(Af) = \ell(Af/rAf(eA)fA) < \infty\), a colocal pair \((eA, Af)\) in \(A\) is called finite.

In [5], a pair \((eA, Af)\) of a right ideal \(eA\) and a left ideal \(Af\) in \(A\) is called an \(i\)-pair if \(e, f \in A\) are local idempotents, \(eA_A\) is colocal with \(\text{soc}(eA_A) \cong fA/fJ\) and \(Af\) is colocal with \(\text{soc}(A_Af) \cong Ae/Je\).

Lemma 5.1. Let \(e, f \in A\) be local idempotents. Then the following are equivalent.

1. \((eA, Af)\) is an \(i\)-pair in \(A\).
2. \((eA, Af)\) is a colocal pair in \(A\) with \(l_{eA}(Af) = 0\) and \(r_{Af}(eA) = 0\).

Proof. (1) \(\Rightarrow\) (2). By (1) and (3) of Lemma 3.1.

(2) \(\Rightarrow\) (1). By Corollary 3.3.

The equivalence (1) \(\Leftrightarrow\) (2) of the next lemma has been established in [5, Theorem 3.7]. Here we provide another proof of the implication (2) \(\Rightarrow\) (1) which does not appeal to Morita duality.

Lemma 5.2 ([5, Theorem 3.7]). Let \((eA, Af)\) be an \(i\)-pair in a left or right perfect ring \(A\). Then the following are equivalent.

1. \((eA, Af)\) is finite.
2. Both \(eA_A\) and \(Af\) are injective.
3. \(eA_A\) is injective and \(Af\) is \(A\)-simple-injective.

Proof. (1) \(\Rightarrow\) (2). By Theorem 4.4.

(2) \(\Rightarrow\) (3). Obvious.

(3) \(\Rightarrow\) (1). It follows by Lemma 3.4(1) that \(X = l_{eA}(rAf(X))\) for every submodule \(X\) of \(eAeA\). Thus by Proposition 4.6 \(\ell(fAfA) < \infty\).

Lemma 5.3. Let \((eA, Af)\) be a finite colocal pair in a left or right perfect ring \(A\). Then the following hold.

1. \(eA/eA(Af)_A\) is a quasi-injective colocal module with \(\text{soc}(eA/eA(Af)_A) \cong fA/fJ\).
2. If \(rAf(eA) = 0\), then \(E(fA/fJ_A) \cong eA/eA(Af)\), so that \(E(fA/fJ_A)\) is quasi-projective and \(eA/eA(Af)_A\) is injective.
Proof. Put $I = l_A(Af)$ and $L = eA/eI_A$. Then $l_{eA}(Af) = eI$ and $l_L(Af) = 0$.

Note that, since $If = 0$, $Lf_{f_AF} \cong eA_{f_AF}$. Thus by Lemma 3.2 $L_A$ is colocal with soc$(L_A) \cong fA/fJ$. Since $Lf_{f_AF} \cong eA_{f_AF}$ and $H = \text{End}_A(L_A) \cong eA/eI$, $H_Lf_{f_AF}$ is a colocal bimodule. Note also that $\ell(Af/\mathcal{F}(Af)(eA)f_{f_AF}) < \infty$.

(1) By Corollary 4.5 $L_A$ is quasi-injective.

(2) By Theorem 4.4 $L_A$ is injective. Thus, since soc$(L_A) \cong fA/fJ$, $E(fA/fJ_A)$
$\cong L$. Since $L_{A/I} \cong e(A/I)_{A/I}$ is projective, $L_A$ is quasi-projective. □

Proposition 5.4. Let $(eA, Af)$ be a colocal pair with $l_{eA}(Af) = 0$ in a left or right perfect ring $A$. Put $A = A/e\mathcal{A}_{A}(eA)$. Let $\pi : A \to \mathcal{A}$ be the canonical ring homomorphism and put $\bar{e} = \pi(e)$, $\bar{f} = \pi(f)$. Then the following are equivalent.

(1) $(eA, Af)$ is finite.

(2) $eA_A$ is quasi-injective, $eA Ae^A$ is finitely generated and $A Af_{f_AF}(eA)$ is injective.

(3) $(\bar{eA}, \bar{Af})$ is a finite $i$-pair in $\mathcal{A}$.

Proof. Note first that $\mathcal{A}$ is left or right perfect and $\bar{e}, \bar{f} \in \mathcal{A}$ are local idempotents. Put $I = r_A(eA)$. Then $I = 0$ and $If = r_A(eA)$. Thus $\ell_{(eAe^A)} = \ell_{eAe^A}$ and, since $eAe^A_{f_AF} \cong eAe^A_{f_AF}$ is a colocal bimodule, $(\bar{eA}, \bar{Af})$ is a colocal pair in $\mathcal{A}$.

(1) $\Rightarrow$ (2). By Lemma 5.3(1) $eA_A$ is quasi-injective, and by Lemma 5.3(2) $A Af_{f_AF}(eA)$ is injective. Also, since $\ell_{(eAe^A)} < \infty$, $eAe^A$ is finitely generated.

(2) $\Rightarrow$ (3). By [3, Corollary 5.6A] $\bar{eA_{A}} \cong eA_{A}$ is injective. Also, since $A Af_{f_AF}(eA)$ is injective, so is $\bar{A Af}_{f_AF}$. It is obvious that $r_{\mathcal{A}}(\bar{eA}) = 0$. For any $a \in l_{eA}(\bar{Af})$, since $aAf \subseteq 1f$, $aAf = eaAf \subseteq eI = 0$ and $a \in l_{eA}(Af) = 0$. It follows that $l_{eA}(\bar{Af}) = 0$. Thus by Lemmas 5.1 and 5.2 $(\bar{eA}, \bar{Af})$ is a finite $i$-pair in $\mathcal{A}$.

(3) $\Rightarrow$ (1). Obvious. □

Corollary 5.5. Let $(eA, Af)$ be an $i$-pair in a left or right perfect ring $A$. Then the following are equivalent.

(1) $(eA, Af)$ is finite.

(2) $eA_A$ is quasi-injective, $eA Ae^A$ is finitely generated and $A Af$ is injective.

6. Applications of colocal pairs I

In this section, as applications of colocal pairs, we extend recent results of Baba [1, Theorems 1 and 2] to left perfect rings and provide simple proofs of them.

Lemma 6.1. Let $A$ be a left perfect ring and $e \in A$ a local idempotent. Assume $AE = E(Aae^A)$ is quasi-projective. Then $Ae/JE$ is simple and for a local idempotent $f \in A$ with $Ae/JE \cong Af/Jf$ the following hold:

...
(a) \( A/E \cong A/rAf(eA) \);
(b) \( eAeAf \cong eAeE \) is injective; and
(c) \( (eA, Af) \) is a colocal pair in \( A \) with \( l_{eA}(Af) = 0 \).

Proof. Put \( I = l_A(E) \). By Lemma 1.4 there exists a local idempotent \( f \in A \) such that \( A/E \cong A/If \). We claim \( If = rAf(eA) \). Since by Lemma 3.5(2) \( eAf = eIf \subseteq l_{eA}(E) = 0 \), \( If \subseteq rAf(eA) \). Conversely, let \( a \in rAf(eA) \). Then \( eA(a+If) = 0 \) and by Lemma 3.1(1) \( (a+If) \in rAf/Af(eA) = 0 \), so that \( a \in If \). Next, since \( e(rAf(eA)) = 0 \), \( eAeE \cong eAe(If/rAf(eA)) \cong eAeAf \). Thus \( eAeAf \) is colocal by Lemma 3.1(3) and injective by Lemma 1.2(2). Also, since \( \text{End}_A(AAf/I) \cong fAf/fIf \), by Lemma 3.5(1) \( AAfAf \) is colocal. Finally, by Lemma 3.5(2) \( l_{eA}(Af) \subseteq l_{eA}(Af/rAf(eA)) = l_{eA}(E) = 0 \).

Theorem 6.2 (cf. [1, Theorem 1]). Let \( A \) be a left perfect ring and \( e, f \in A \) local idempotents. Put \( E = E(Ae/Je) \). Assume \( \ell(Af/rAf(eA)fAf) < \infty \). Then the following are equivalent.

1. \( eA \) is quasi-injective with \( \text{soc}(eA) \cong fA/fJ \).
2. \( A/E \) is quasi-projective with \( A/E/Je \cong Af/Jf \).
3. \( (eA, Af) \) is a colocal pair in \( A \) with \( l_{eA}(Af) = 0 \).
4. \( eAeAf \) is colocal and \( \text{soc}(eA) \cong fA/fJ \).

Proof. (1) \( \Rightarrow \) (3). By Lemma 3.5(1).
(3) \( \Rightarrow \) (1). By Lemma 5.3(1).
(2) \( \Rightarrow \) (3). By Lemma 6.1.
(3) \( \Rightarrow \) (2). By Lemma 5.3(2).
(3) \( \Rightarrow \) (4). By Corollary 3.3.
(4) \( \Rightarrow \) (3). By (3) and (1) of Lemma 3.1.

Lemma 6.3. Let \( (eA, Af) \) be a colocal pair in a left or right perfect ring \( A \). Put \( E = E(Ae/Je) \) and \( H = \text{End}_A(AE)^\circ \). Assume \( \text{soc}(eA) \neq 0 \). Then the following hold.

1. \( \text{soc}(eA)fA \) is the unique simple submodule of \( eA \) which is isomorphic to \( fA/fJ \).
2. If \( (eA, Af) \) is finite, then \( A/EH \) contains a submodule \( X \) such that \( A/X \cong Af/rAf(eA)fAf, eAeXH \) is a colocal bimodule, \( \text{soc}(eA)fA \cap l_{eA}(X) = 0 \) and \( \ell(eAeA/l_{eA}(X)) < \infty \).

Proof. (1) Since \( \text{soc}(eA)f \neq 0, eA \) contains a simple submodule \( K \cong fA/fJ \). On the other hand, by Corollary 3.3 \( eA/l_{eA}(Af) \) is colocal with \( \text{soc}(eA/l_{eA}(Af)) \cong fA/fJ \). Thus \( K \) is the unique simple submodule of \( eA \) which is isomorphic to \( fA/fJ \). It follows that \( K = \text{soc}(eA)fA \).
(2) Put $I = r_A(eA)$. Then $If = r_Af(eA)$ and by Lemma 5.3(1) $A Af/If$ is a quasi-injective colocal module with $soc(A Af/If) \cong Ae/Je$. Thus $AE$ contains a submodule $X \cong A Af/If$. Then $Af/If = r_Af(eA)$ and by Lemma 5.3(1) $A Af/If$ is a quasi-injective colocal module with $soc(A Af/If) = Ae/Je$. Thus $A E$ contains a submodule $X = A Af/If$. Then by Lemma 1.3 $XH \subset X$. Since by Lemma 3.5(1) $eAe eH$ is a colocal bimodule, so is $eAe XH$. Also, since $eI = 0$, $soc(eA)fA(Af/If) \cong soc(eA)AfAf \neq 0$. Thus $soc(eA)fA \cap l_{eA}(X) = 0$. Finally, since $l_A(X) = l_{eA}(Af)$, $\ell(eAeA/eA(eA(Af))) < \infty$.

Lemma 6.4. Let $A$ be a left perfect ring and $e \in A$ a local idempotent. Put $E = E(A Ae/Je)$ and $H = \text{End}_{eA}(AE)^{op}$. Assume $soc(eA) \cong \bigoplus_{i=1}^n f_i A/f_i J$ with the $(eA, A f_i)$ finite colocal pairs in $A$. Then $f_i A/f_i J \neq f_j A/f_j J$ for $i \neq j$, $\ell(EH) = \ell(eAeA) < \infty$ and $AE/JE \cong \bigoplus_{i=1}^n A f_i /J f_i$.

Proof. By Lemma 6.3(1) $f_i A/f_i J \neq f_j A/f_j J$ for $i \neq j$. Also, for each $1 \leq i \leq n$, by Lemma 6.3(2) $A eH$ contains a submodule $X_i$ such that $A X_i \cong A f_i/r_A(eA)f_i$, $eAe eX_i H$ is a colocal bimodule, $soc(eA)f_i A \cap l_{eA}(X_i) = 0$ and $\ell(eAeA/eA(eA(X_i))) < \infty$. Put $A X_H = \sum_{i=1}^n X_i$. Then, by Lemmas 3.1(1) and 2.5 $\ell(X_H) = \ell(eAeA/eA(eA(X_i))) < \infty$ for all $1 \leq i \leq n$, so that $\ell(X_H) < \infty$. Also, since $soc(eA)f_i A \cap l_{eA}(X_i) = 0$ for all $1 \leq i \leq n$, by Lemma 6.3(1) $soc(eA) \cap l_{eA}(X) = 0$. Thus, since $eAe$ has essential socle, $l_{eA}(X) = 0$. Since by Lemma 3.5(1) $eAe eXH$ is a colocal bimodule, so is $eAe eX_H$. Thus by Lemma 2.5 $\ell(eAeA/eA(Af)) = \ell(X_H) < \infty$. Since by Lemma 1.3 we have a surjective ring homomorphism $\rho_X : H \rightarrow \text{End}_{eA}(AX)^{op}$, $h \mapsto h|X$, it follows by Theorem 4.4 that $AX$ is injective. Thus $X = E$ and we have an epimorphism $\bigoplus_{i=1}^n f_i A /f_i J \rightarrow AE/JE$. On the other hand, since $f_i A/f_i J \neq f_j A/f_j J$ for $i \neq j$, it follows by Lemma 3.5(2) that $AE/JE$ has a direct summand which is isomorphic to $\bigoplus_{i=1}^n A f_i /J f_i$. Thus $AE/JE \cong \bigoplus_{i=1}^n A f_i /J f_i$.

Theorem 6.5 (cf. [1, Theorem 2]). Let $A$ be a left perfect ring and $e, f_1, f_2, \ldots, f_n \in A$ local idempotents. Put $E = E(A Ae/Je)$. Assume $(eA, A f_i)$ is a finite colocal pair in $A$ for all $1 \leq i \leq n$. Then the following are equivalent.

1. $soc(eA) \cong \bigoplus_{i=1}^n f_i A/f_i J$.
2. $AE/JE \cong \bigoplus_{i=1}^n A f_i /J f_i$.

Proof. (1) $\Rightarrow$ (2). By Lemma 6.4.

(2) $\Rightarrow$ (1). It follows by Lemmas 3.5(2) and 6.3(1) that $soc(eA)$ is isomorphic to a direct summand of $\bigoplus_{i=1}^r f_i A/f_i J$. We may assume $soc(eA) \cong \bigoplus_{i=1}^r f_i A/f_i J$ for some $1 \leq r \leq n$. Then by Lemma 6.4 $AE/JE \cong \bigoplus_{i=1}^r A f_i /J f_i$, so that $r = n$.

7. Applications of colocal pairs II

In this section, we provide some other applications of colocal pairs. Recall that a set $\{e_1, \cdots, e_n\}$ of orthogonal local idempotents in a semiperfect ring $A$ is called
basic if \((\sum_{i=1}^{n} e_i)A(\sum_{i=1}^{n} e_i)\) is a basic ring of \(A\).

**Lemma 7.1** ([5, Lemma 3.5]). Let \(A\) be a semiperfect ring and \(\{e_1, \cdots, e_n\}\) a basic set of orthogonal local idempotents in \(A\). Assume every \(e_iA_A\) is \(A\)-simple-injective and has essential socle. Then there exists a permutation \(\nu\) of the set \(\{1, \cdots, n\}\) such that \((e_iA_A, Ae_{\nu(i)})\) is an \(i\)-pair in \(A\) for all \(1 \leq i \leq n\).

**Proof.** By [5, Lemma 3.5] there exists a mapping \(\nu : \{1, \cdots, n\} \to \{1, \cdots, n\}\) such that \((e_iA, Ae_{\nu(i)})\) is an \(i\)-pair in \(A\) for all \(1 \leq i \leq n\). Then by the definition of \(i\)-pairs \(\nu\) is injective.

**Corollary 7.2.** Let \(A\) be a left perfect ring. Assume \(A_A\) is simple-quasi-injective. Then \(E(AA)\) and \(E(A_A)\) are injective cogenerators in \(\text{Mod} \ A\) and \(\text{Mod} \ \text{A}^{\text{op}}\), respectively.

**Lemma 7.3.** Let \(A\) be a left perfect ring. Assume \(A_r(A, A)\) satisfies the ACC and \(eA_A\) is simple-quasi-injective for every local idempotent \(e \in A\). Then \(A\) is left artinian.

**Proof.** It suffices to show that \(\ell_{(eAeA)} < \infty\) for every local idempotent \(e \in A\). Let \(e \in A\) be a local idempotent. Since by Lemma 3.6 \(eA_A\) is colocal, there exists a local idempotent \(f \in A\) with soc\((eA_A) \cong fA/fJ\). By Lemma 3.5(1) \((eA, Af)\) is a colocal pair in \(A\) with \(l_{eA}\)(Af) = 0. For each \(M \in A_r(eA, Af)\), put \(M = r_{A}(l_{eA}(M)) \in A_r(A, A)\). Then 
\[
\tilde{M}f = r_{Af}(l_{eA}(M)) = M \quad \text{for every} \quad M \in A_r(eA, Af).
\]
Thus, for \(M, N \in A_r(eA, Af)\) with \(M \subset N, M \subset \tilde{N}\) and \(\tilde{M} = \tilde{N}\) implies \(M = \tilde{M}f = \tilde{N}f = N\). It follows that \(A_r(eA, Af)\) satisfies the ACC. Thus by Lemmas 2.5 and 2.6 \(\ell_{(eAeA)} = \ell(Af/r_{Af}(eA)_{fAf}) < \infty\).

**Corollary 7.4.** Let \(A\) be a left perfect ring. Assume \(A_r(A, A)\) satisfies the ACC and \(A_A\) is simple-quasi-injective. Then \(A\) is quasi-Frobenius.

**Proof.** By Lemma 7.3 \(A\) is left artinian. Then it follows by Lemmas 3.6 and 4.1 that \(A_A\) is injective.

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