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Author: Sugitani, Sadao

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ON NONEXISTENCE OF GLOBAL SOLUTIONS FOR SOME NONLINEAR INTEGRAL EQUATIONS

SADAO SUGITANI

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1. Statement of the problem

Let \( a(x) \) be a nonnegative continuous function defined on the \( m \)-dimensional Euclidean space \( \mathbb{R}^m \) and let \( \Delta \) be the Laplacian. Consider the following semilinear parabolic equation

\[
\frac{\partial u}{\partial t} = \frac{1}{2} \Delta u + u^{1+\sigma},
\]

with the initial condition

\[
u(0, x) = a(x),
\]

and be concerned with non-negative solutions.

H. Fujita [1] has proved that equation (1.1) has a global solution \( u(t, x) \) for sufficiently small \( a(x) \) when \( ma > 2 \) but (1.1) has no global solution for any \( \phi \) \( \neq 0 \) when \( ma < 2 \). Recently, K. Hayakawa [2] has proved that (1.1) has no global solution even in the critical case \( ma = 2 \) if the dimension \( m \) equals 1 or 2 (and hence \( \alpha = 2 \) or 1, respectively).

In this paper we shall treat this kind of blowing-up problem for a more general equation as follows. Let \( 0 < \alpha \leq \beta \leq 2 \). Let \( F(u) \) be a nonnegative continuous function with \( F(0) = 0 \), defined on \([0, \infty)\), satisfying the following conditions:

(F.1) \( F \) is increasing and convex.

(F.2) There exists some \( \alpha \in \left[0, \frac{\beta}{m}\right] \) and \( c' \in (0, \infty) \), such that

\[
\lim_{u \to 0} \frac{F(u)}{u^{1+\sigma}} = c'.
\]

(F.3) \[
\int_0^{\infty} \frac{du}{F(u)} < \infty.
\]

It is obvious that, for \( 0 < ma \leq \beta \), \( u^{1+\sigma} \) satisfies the above conditions.

Here and hereafter, \( u \) denotes a single variable as well as function in obvious
contexts.

For $0 < \beta \leq 2$, let $\left(-\frac{\Delta}{2}\right)^{\beta/2}$ denote the fractional power of the operator $-\frac{\Delta}{2}$. As a generalization of (1.1), we consider the equation

\begin{equation}
\begin{cases}
\frac{\partial u}{\partial t} = -\left(-\frac{\Delta}{2}\right)^{\beta/2} u + F(u), \\
u(0, x) = a(x).
\end{cases}
\end{equation}

Let $p(t, x)$ be the fundamental solution of (1.2) for $F(u) = 0$, i.e., the density of the semigroup of $(m$-dimensional) symmetric stable process with index $\beta$. It is well known that $p(t, x)$ is given by

\begin{equation}
\int_{\mathbb{R}^m} e^{ix \cdot y} p(t, x) dx = e^{-t|y|^\beta} 0 < \beta \leq 2.
\end{equation}

Using this $p(t, x)$, we can transform (1.2) into the integral equation

\begin{equation}
\begin{aligned}
u(t, x) &= \int_{\mathbb{R}^m} p(t, x-y)a(y)dy + \int_0^t ds \int_{\mathbb{R}^m} p(t-s, x-y)F[u(s, y)]dy, \\
&\quad t > 0, \quad x \in \mathbb{R}^m.
\end{aligned}
\end{equation}

What we are going to prove is the following.

**Theorem.** Let $0 < \beta \leq 2$. Suppose that $a(x)$ is a nontrivial ($\neq 0$), nonnegative, and continuous function on $\mathbb{R}^m$, that $F(u)$ satisfies (F.1), (F.2), (F.3), and that $p(t, x)$ is defined by (1.3). Then the nonnegative solution $\nu(t, x)$ of the integral equation (A) blows up, i.e., there exists some $t_0 > 0$ such that $\nu(t, x) = \infty$ for every $t \geq t_0$ and $x \in \mathbb{R}^m$.

2. Some properties of $p(t, x)$

We here collect some properties of $p(t, x)$ which are required to show our Theorem. By (1.3), we have

\begin{equation}
p(t, x) = t^{-m/\beta} p(1, t^{-1/\beta} x),
\end{equation}

\begin{equation}
p(ts, x) = t^{-m/\beta} p(s, t^{-1/\beta} x).
\end{equation}

Note that $p(t, 0)$ is a decreasing function of $t$. It is known (see [3; pp. 259–268.]) that

\begin{equation}
p(t, x) = \int_0^\infty f_{t, \beta/2}(s) T(s, x) ds \quad \text{ for } 0 < \beta < 2,
\end{equation}

\begin{equation}
T(t, x) \quad \text{ for } \beta = 2,
\end{equation}

where

\begin{equation}
f_{t, \beta/2}(s) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} e^{zs} t^{-\beta/2} dz \geq 0, \sigma > 0, s > 0,
\end{equation}

\begin{equation}
0 < \beta < 2.
\end{equation}
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\[ T(s, x) = \left(\frac{1}{2\pi s}\right)^{m/2} \exp\left(-\frac{|x|^2}{2s}\right). \]

The above relation implies that \( p(t, x) \) is a decreasing function of \( |x| \), i.e.,

\[ p(t, x) \leq p(t, y) \quad \text{whenever } |x| \geq |y|. \]  

(2.3)

We sometimes write \( p(t, |x|) \) for \( p(t, x) \). Combining (2.1) and (2.3),

\[ p(t, x) \leq \left(\frac{x}{y}\right)^{m/\beta} p(s, x) \quad \text{for } t \geq s. \]  

(2.4)

Finally, it follows that

\[ p(t, 0) \leq 1 \quad \text{and } \tau \geq 2, \text{ then } \quad p\left(t, \frac{1}{\tau} (x - y)\right) \geq p(t, x)p(t, y). \]  

(2.5)

Because \( \frac{1}{\tau} |x - y| \leq \frac{2}{\tau} |x| \lor \frac{2}{\tau} |y| \leq |x| \lor |y| \), and hence \( p\left(t, \frac{1}{\tau} (x - y)\right) \geq p(t, |x| \lor |y|) \geq p(t, x) \land p(t, y) \geq p(t, x)p(t, y). \)

3. Preliminary lemmas

Lemma 1. If \( F \) satisfies (F.1) and (F.3), then

\[ \lim_{u \to \infty} \frac{1}{u} F(u) = \infty. \]  

(3.1)

Proof. Since \( F \) is convex, it is obvious that \( \frac{1}{u} (F(u) - F(0)) \) is a monotone increasing function. If \( \lim_{u \to \infty} \frac{1}{u} (F(u) - F(0)) = M < \infty \), then \( \frac{1}{u} (F(u) - F(0)) \leq M \) for all \( u > 0 \), i.e., \( \frac{1}{Mu} \leq \frac{1}{F(u) - F(0)}. \) This contradicts assumption (F.3).

If \( F(u) \) is increasing, \( F(\infty) \) is defined by

\[ F(\infty) = \lim_{u \to \infty} F(u) \leq \infty. \]  

(3.2)

Lemma 2. (Jensen's inequality) Let \( \rho \) be a probability measure on \( R^n \) and \( u(x) \) a nonnegative function. Suppose that \( F(u) \) satisfies (F.1). Then we have

\[ F\left(\int_{R^n} u d\rho\right) \leq \int_{R^n} F \circ u d\rho. \]  

(3.3)

Note that this inequality is valid even when \( \int_{R^n} u d\rho = \infty. \)

Lemma 3. Suppose that \( F(u) (\neq 0) \) satisfies (F.1). Let \( u(t, x) \) be a nonnegative solution of (A) and let

\[ f(t) = \int_{R^n} p(t, x) u(t, x) dx. \]  

(3.4)
Then the following two conditions are equivalent:
(a) \( u(t, x) \) blows up.
(b) \( f(t) \) blows up, i.e., there exists some \( t_i > 0 \) such that \( f(t) = \infty \) whenever \( t \geq t_i \).

Proof. It is enough to show that (b) implies (a). We may assume \( p(t_i, 0) \leq 1 \), so that \( p(t, 0) \leq 1 \) for any \( t \geq t_i \). If \( t_i \leq t, t \leq \frac{8}{2^p+1} t_i \), then

\[
p(8t-s, x-y) = p\left(s \left(\frac{8t-s}{s}\right), x-y\right) = \left(\frac{s}{8t-s}\right)^{\frac{m}{\beta}} p\left(s, \left(\frac{s}{8t-s}\right)^{\frac{1}{\beta}} (x-y)\right) \quad \text{by (2.1)}
\]

\[
\geq \left(\frac{s}{8t-s}\right)^{\frac{m}{\beta}} p(s, x)p(s, y) \quad \text{by (2.5)}.
\]

Therefore,

\[
\int_{R^m} p(8t-s, x-y)u(s, y)dy \geq \left(\frac{s}{8t-s}\right)^{\frac{m}{\beta}} p(s, x)f(s) = \infty.
\]

Finally, applying Jensen's inequality to \((A)\) and noting that \( F(\infty) = \infty \), we have

\[
u(8t, x) \geq \int_{8t}^{t} dsF\left[\int_{R^m} p(8t-s, x-y)u(s, y)dy\right] = \infty,
\]

so that \( u(t, x) = \infty \) for any \( t \geq 8t_i \), and \( x \in R^m \).

4. Proof of the theorem

Let \( u(t, x) \) be a nonnegative solution of \((A)\), then we can find \( t_o > 0, c > 0, \)
\( \gamma > 0 \) such that \( u(t_o, x) \geq cp(\gamma, x) \). In fact, if we choose \( t_o > 0 \) such that \( p(t_o, 0) \leq 1 \), we have

\[
p(t_o, x-y) = p\left(t_o, \frac{1}{2} (2x-2y) \right)
\]

\[
\geq p(t_o, 2x)p(t_o, 2y) \quad \text{by (2.5)}
\]

\[
= 2^{-m}p\left(\frac{t_o}{2^p}, x\right)p(t_o, 2y) \quad \text{by (2.2)}.
\]

Therefore, \( u(t_o, x) \geq \int_{R^m} p(t_o, 2y)du(y)dy \cdot 2^{-m}p\left(\frac{t_o}{2^p}, x\right) \). But \( u(t+t_o, x) \) satisfies

\[
u(t+t_o, x) = \int_{R^m} p(t, x-y)u(t_o, y)dy
\]

\[
+ \int_{t}^{t+t_o} dt \int_{R^m} p(t-s, x-y)F[u(s+t_o, y)]dy
\]

\( t > 0, x \in R^m \),

so that
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(4.2) \[ u(t+t_0, x) \geq c(t+\gamma, x) + \int_0^t ds \int_{\mathbb{R}^m} p(t-s, x-y)F[u(s+t_0, y)]dy. \]

Hence, by the comparison theorem, it is enough to show that the solution \( v(t, x) \) of the equation

(B) \[ v(t, x) = c(t+\gamma, x) + \int_0^t ds \int_{\mathbb{R}^m} p(t-s, x-y)F[v(s, y)]dy \]

blows up, or by virtue of Lemma 3, that

(4.3) \[ f(t) = \int_{\mathbb{R}^m} p(t, x)v(t, x)dx \]

blows up. Multiplying both sides of (B) by \( p(t, x) \), and integrating, we have

(4.4) \[ f(t) = c(2t+\gamma, 0) + \int_0^t ds \int_{\mathbb{R}^m} p(2t-s, y)F[v(s, y)]dy \]

\[ \geq c(1, 0)(2t+\gamma)^{-m/\beta} + \int_0^t ds \left( \frac{s}{2(t-s)} \right)^{m/\beta} \int_{\mathbb{R}^m} p(s, y)F[v(s, y)]dy \]

(by (2.1), (2.4))

\[ \geq c(1, 0)(2t+\gamma)^{-m/\beta} + \int_0^t ds \left( \frac{s}{2t} \right)^{m/\beta} F[f(s)] \]

(by Jensen’s inequality)

Let \( \delta > 0 \) be a fixed positive constant. Hereafter we always assume \( t \geq \delta \).

Put \( f_1(t) = f^{m/\beta} f(t) \), then by (4.4),

(4.5) \[ f_1(t) \geq c(1, 0) \left( \frac{\delta}{2\delta+\gamma} \right)^{m/\beta} + \int_0^t ds \left( \frac{s}{2t} \right)^{m/\beta} F[f_1(s)s^{-m/\beta}] . \]

Let \( f_2(t) \) be the solution of

(4.6) \[ f_2(t) = c(1, 0) \left( \frac{\delta}{2\delta+\gamma} \right)^{m/\beta} + \int_0^t ds \left( \frac{s}{2t} \right)^{m/\beta} F[f_2(s)s^{-m/\beta}] . \]

By assumption (F.2) and Lemma 1, there exists \( a > 0 \) such that

\[ \max \left( \frac{F(u)}{u} , \frac{F(u)}{u^{1+a}} \right) \geq a \text{ for all } u > 0. \]

Since

\[ s^{m/\beta} F(f_2(s)s^{-m/\beta}) = \frac{F(f_2(s)s^{-m/\beta})}{f_2(s)s^{-m/\beta}} \cdot f_2(s) = \frac{F(f_2(s)s^{-m/\beta})}{(f_2(s)s^{-m/\beta})^{1+a}} \cdot f_2(s)^{1+a}s^{-m/\beta} \]

it follows that

\[ s^{m/\beta} F(f_2(s)s^{-m/\beta}) \geq a \cdot \min (f_2(s), f_2(s)^{1+a}s^{-m/\beta}). \]

Therefore,
Let \( f_3(t) \) be the solution of the integral equation

\[
(4.7) \quad f_3(t) = cp(1, 0) \left( \frac{\delta}{2\delta + \gamma} \right)^{m/\beta} + \int_{0}^{t} ds \left( \frac{1}{2} \right)^{m/\beta} a \cdot \min \left( f_3(s), f_3(s)^{1+\alpha} s^{-m/\beta} \right),
\]

or, equivalently, the ordinary differential equation

\[
(4.8) \quad \begin{cases}
\frac{df_3(t)}{dt} = \left( \frac{1}{2} \right)^{m/\beta} a \cdot \min \left( f_3(t), f_3(t)^{1+\alpha} t^{-m/\beta} \right), \\
f_3(0) = cp(1, 0) \left( \frac{\delta}{2\delta + \gamma} \right)^{m/\beta}.
\end{cases}
\]

We shall show that \( f_3(t) \) increases exponentially fast. This is obvious if \( \alpha = 0 \). Next we consider the case \( \alpha > 0 \). By the comparison theorem, \( c \) can be chosen arbitrarily small. We choose \( c \), if necessary, satisfying the following three conditions (4.9), (4.10) and (4.11).

\[
(4.9) \quad f_3(\delta) < \delta^{m/\beta}.
\]

Put \( \theta(c) = \inf \{t \geq \delta; f_3(t) = t^{m/\beta}\} \). For \( t \in [\delta, \theta(c)] \), \( \min \left\{ f_3(t), f_3(t)^{1+\alpha} t^{-m/\beta} \right\} = f_3(t)^{1+\alpha} t^{-m/\beta} \) by (4.9). Therefore, \( f_3(t) \) satisfies the equation \( \frac{df_3(t)}{dt} = \left( \frac{1}{2} \right)^{m/\beta} a f_3(t)^{1+\alpha} t^{-(m/\beta)} \), which implies that \( \theta(c) < \infty \). (We here use the condition \( m\alpha \leq \beta \) in (F.2)). On the other hand \( \lim_{c \to 0} \theta(c) = \infty \). Hence, if \( c \) is small enough, we have

\[
(4.10) \quad \exp \left[ \left( \frac{1}{2} \right)^{m/\beta} a \frac{t}{\theta(c)^{m/\beta}} \right] t \geq \theta(c),
\]

\[
(4.11) \quad \theta(c)^{-m/\beta} \leq \alpha \left( \frac{1}{2} \right)^{m/\beta} a \int_{0}^{t} s^{-m/\beta} ds \leq \theta(c). \}
\]

For \( t \geq \theta(c) \),

\[
\begin{cases}
f_3(\theta(c)) = \theta(c)^{m/\beta}, \\
\frac{df_3(t)}{dt} = \left( \frac{1}{2} \right)^{m/\beta} a \cdot \min \left( f_3(t), f_3(t)^{1+\alpha} t^{-m/\beta} \right).
\end{cases}
\]

Let \( x_1(t) \) and \( x_2(t) \) be the solutions of the following equations;

\[
(4.12) \quad \begin{cases}
x_1(\theta(c)) = \theta(c)^{m/\beta}, \\
\frac{dx_1}{dt} = \left( \frac{1}{2} \right)^{m/\beta} a x_1,
\end{cases}
\]
Then it follows that, for \( t \geq \theta(c) \), \( x(t) \geq t^{m/\beta} \) by (4.10) and \( x(t) \geq t^{m/\beta} \) by (4.11). From this, it is not difficult to see that \( f_3(t) = x_1(t) \) for \( t \geq \theta(c) \). Thus \( f_3(t) \) increases exponentially fast. Hence there exists \( b > 0 \) such that
\[
(4.14) \quad f_3(t) \geq be^{bt}.
\]
By the comparison theorem, \( f_1 \geq f_2 \geq f_3 \geq be^{bt} \). Put \( h(t) = t^{-m/\beta}f_3(t) \). Then, since \( f(t) \geq h(t) \), it is sufficient to show that \( h(t) = \infty \) if \( t \) is large enough. Suppose that \( h(t) < \infty \) for every \( t > \delta \). Noting that \( h(t) \to \infty \) as \( t \to \infty \) and using Lemma 1, we have
\[
(4.15) \quad \sup_{t \geq t'} \frac{m}{\beta t} \frac{h(t)}{F(h(t))} \leq \left( \frac{1}{2} \right)^{m/\beta + 1} \text{ for some } t' > 0.
\]
By (4.6), (4.15), we have for \( t \geq t' \)
\[
\frac{dh(t)}{dt} = -\frac{m}{\beta t} t^{-m/\beta} f_3(t) + t^{-m/\beta} \frac{df_3(t)}{dt} = -\frac{m}{\beta t} t^{-m/\beta} f_3(t) + \left( \frac{1}{2} \right)^{m/\beta} F(f_3(t)t^{-m/\beta}) = \left( \frac{1}{2} \right)^{m/\beta} F(h(t)) - \frac{m}{\beta t} h(t) \geq \left( \frac{1}{2} \right)^{m/\beta + 1} F(h(t)).
\]
It then follows that
\[
\left( \frac{1}{2} \right)^{m/\beta + 1} (t - t') \leq \int_{t' \wedge t}^{h(t')} \frac{dx}{F(x)} \leq \int_{t' \wedge t}^{\infty} \frac{dx}{F(x)} < \infty
\]
for any \( t \geq t' \), which is a contradiction.