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SINGULARITIES OF 2-SPHERES IN 4-SPACE AND COBORDISM OF KNOTS[1)]

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Consider an oriented 2-dimensional manifold m imbedded as a subcomplex in a triangulated oriented 4-dimensional manifold M in such a way that the boundary of m is contained in the boundary of M and the interior of m is contained in the interior of M. We will assume that M is a "piecewise linear manifold": that is, the star neighborhood of any point should be piecewise linearly homeomorphic to a 4-simplex. One can measure the local singularity of the imbedding at an interior point x of m as follows. Let N denote the star neighborhood of x in M. The boundary $S=\partial N$ of N is a 3-sphere with an orientation inherited from that of M, and $k=m\cap\partial N$ is a 1-sphere with an orientation inherited from that of m. The oriented knot type κ of the imbedding of k in S is called[2)] the *singularity* of the imbedding at x. When k is of trivial type in ∂N we may say that *the singularity is* 0 or that x is a *non-singular point* or that m is *locally flat* at x. A surface m is called *locally flat* if it is locally flat at each of its points.

REMARK. The singularity of m at x is clearly a combinatorial invariant of M,m,x; that is it is not altered if we subdivide M rectilinearly. We do not know whether or not this singularity is a topological invariant, except in the special case of a locally flat point. The topological invariance of the concept of local flatness is easily proved, making use of Dehn's lemma, [12, §28(i)].

Of course the local singularity can also be measured at a boundary point x. In this case ∂N is a 3-cell, $m\cap\partial N$ is a 1-cell spanning it, and the singularity is a type of spanning 1-cell. In this paper we shall consider only imbeddings whose boundary points are all non-singular.

Since a singular point must be a vertex in any triangulation of the pair $m\subset M$ the singular points are always isolated. If m is compact (as it will be from now on) there can therefore be only a finite number of singular points. For the rest of this paper m will be a 2-sphere and M will be the 4-dimensional euclidean space R^4; that is, the 4-sphere punctured at ∞. The basic problem

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1) This paper follows our announcement [3]. We wish to express our thanks to C.H. Giffen for help in the revision.

2) These concepts are due to V.K.A. Guggenheim [5, §7. 32].

that motivated this paper is the following: *Under what conditions can a given collection of knot types* $\kappa_1, \cdots, \kappa_n$ *be the set of singularities of some imbedding of a 2-sphere m in the 4-space* R^4*?*

Recall that the various types of knots are the elements of a commutative semigroup[3] $\mathcal{A}$; the operation of this semigroup, which has been variously designated "product", "sum", "composition", etc., will be denoted here simply by the symbol $+$. In section one we show that *a collection* $(\kappa_1, \cdots, \kappa_n)$ *can occur as the set of singularities of some imbedding if and only if the collection consisting of the single element* κ, *where* $\kappa = \kappa_1 + \kappa_2 + \cdots + \kappa_n$, *occurs as the set of singularities of some imbedding.* This reduces the basic problem to the following special case: *Which knot types* κ *can occur as the only singularity of a 2-sphere m in* R^4*?* It is shown that a given κ can occur if and only if there is a locally flat 2-sphere m and a hyperplane J of R^4, which cuts m in two, such that $k = m \cap J$ is a knot of type κ in J. Such a knot $k \subset J$ has been called a *slice knot* and its type κ may be called a *slice type* (Compare [4, p. 135].) Clearly k is a slice knot if and only if it spans a non-singular 2-disk which lies completely within one of the two half-spaces bounded by J.

An example of a slice knot is illustrated in Figure 1. Depending on the number of twists, this figure can represent the knot type 6_1 or 8_{20} or 9_{46}, etc.. (The notation for knot types follows [13, p. 70]. For a proof that such a diagram represents a slice knot see [4, p. 172].)

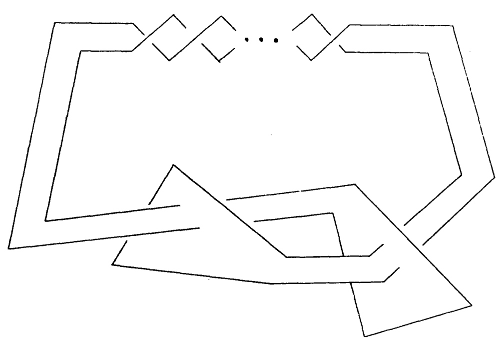

Figure 1.

Our basic question can now be reformulated as follows: *Which knot types are slice types?*

Although it is unreasonable to expect a complete and meaningful answer to this question, partial answers of significance can be looked for. In section two

3) H. Schubert [14]. The semigroup $\mathcal{A}$ is free commutative with the "prime" knot types as free generators.

it is shown that not every knot is a slice knot, inasmuch as[4] *the Alexander polynomial* $A(t)$ *of a slice knot must be of the form* $p(t)\cdot p(1/t)$ *for some integral polynomial* $p(t)$.

As examples, consider the knots with seven or fewer crossings in the Alexander-Briggs table. The Alexander polynomials of these knots (see [1, p. 305]) are all distinct and, with one exception, are all irreducible. Hence these knots cannot be slice knots. The one exception is the stevedore's knot 6_1, with polynomial

$$2-5t+2t^2 = (2-t)(1-2t)\,.$$

We have already remarked that 6_1 is actually a slice knot.

In the third section it is shown that the sum $\kappa+(-\kappa)$ of a knot type κ and the type $-\kappa$ obtained from κ by reversing the orientation of both the knot k and the containing 3–sphere S is always a slice knot. This result makes possible the introduction of an abelian group $\mathcal{G}$ whose elements are equivalence classes $\langle\kappa\rangle$ of knot types κ and whose operation $+$ is inherited from the operation $+$ of the abelian semigroup $\mathcal{A}$. When the equivalence relation $\sim$ that repartitions the elements of $\mathcal{A}$ into elements of $\mathcal{G}$ is expressed in a more symmetrical form which we call *cobordism* it becomes evident that $\mathcal{G}$ is in fact a (relative) cobordism group. In terms of this group the principal results of this paper as well as various outstanding problems may be clearly expressed.

1. Confluence of singularities

Consider a polyhedral 2-sphere m in the 4-space R^4, with singular points $x_1, \cdots, x_x$. Let $\kappa(x_1), \cdots, \kappa(x_n)$ be the corresponding singularity types.

Theorem 1. *The sum* $\kappa(x_1)+\cdots+\kappa(x_n)$ *of the singularities is the knot type of a slice knot.*

Proof. Choose a polygonal arc $p\subset m$ which traverses all of the singular points x_i. Choose some fixed rectilinear triangulation of R^4 so that m and p are subcomplexes. Using this triangulation, let $y_1, \cdots, y_r$ be the vertices of the subcomplex p, listed in their natural order along p. Clearly each singular point x_i occurs as one of these vertices y_j.

Let N denote the star neighborhood of p in the first derived complex of R^4, and let N_j denote the star neighborhood of the vertex y_j; so that

$$N = N_1 \cup N_2 \cup \cdots \cup N_r\,.$$

4) Since this polynomial condition was announced by us in 1957 several other necessary conditions have been established: [10].

Each N_j is a 4-cell and can be identified with the cone over the 3-sphere ∂N_j. Similarly the intersection $m \cap N_j$ is a 2-cell, and can be identified with the cone over $m \cap \partial N_j$. The knotted circle

$$m \cap \partial N_j \subset \partial N_j$$

represents the knot type of the singularity $\kappa(y_j)$.

Note that each intersection $N_j \cap N_{j+1} = \partial N_j \cap \partial N_{j+1}$ is a 3-cell spanned by the unknotted arc $m \cap N_j \cap N_{j+1}$. The cells N_j are mutually disjoint otherwise. From this it follows that their union N is a 4-cell. Furthermore, the circle

$$m \cap \partial N \subset \partial N$$

represents the knot type of the sum $\kappa(y_1) + \cdots + \kappa(y_r)$. This is of course equal to $\kappa(x_1) + \cdots + \kappa(x_n)$.

Choose a base point x_0 on ∂N which does not belong to m. Choose a piecewise linear homeomorphism h from the sphere $S^4 = R^4 \cup \infty$ to itself which carries x_0 to the point at infinity, and carries $\partial N - x_0$ onto the hyperplane J. Then the image $h(m \cap \partial N)$ will be a knot $k \subset J$ representing the required knot type $\kappa(x_1) + \cdots + \kappa(x_n)$. Furthermore $h(m - \text{Interior } N)$ will be a non-singular 2-disk which spans k, and otherwise lies completely on one side of J. Taking the union of this disk with its mirror image in J we obtain a non-singular 2-sphere m' which intersects J in the required knot k. This shows that k is a slice knot, and completes the proof.

REMARK. It is of course essential that m should be a 2-sphere. Any knot of genus one can appear as the unique singularity type of a knotted *torus* in 4 space. Similarly it is essential that the containing 4-manifold should be a sphere or cell. In the (4-dimensional) complex projective plane, any torus knot of type p, $p+1$ can appear as the unique singularity type of an imbedded 2-sphere. (Compare [7]: or consider the algebraic variety which is defined by the homogeneous equation $z_0 z_1^p = z_2^{p+1}$.)

Now consider the converse situation:

Theorem 1′. *Let* $\kappa_1, \cdots, \kappa_r$ *be knot types such that* $\kappa_1 + \cdots + \kappa_n$ *is a slice type. Then there exists a 2-sphere* $m \subset R^4$ *with singularities of type* $\kappa_1, \cdots, \kappa_n$, *and with no other singularities.*

Proof. Represent the knot types $\kappa_1, \cdots, \kappa_n$ by knots $k_1, \cdots, k_n$ which lie within disjoint cubes in the hyperplane $J \subset R^4$, and which can be joined by rectangular bands $B_1, \cdots, B_{n-1} \subset J$ as illustrated in Figure 2. Choose vertices $v_1, \cdots, v_n$ below the hyperplane J, so that the cones

$$v_1 k_1, \cdots, v_n k_n$$

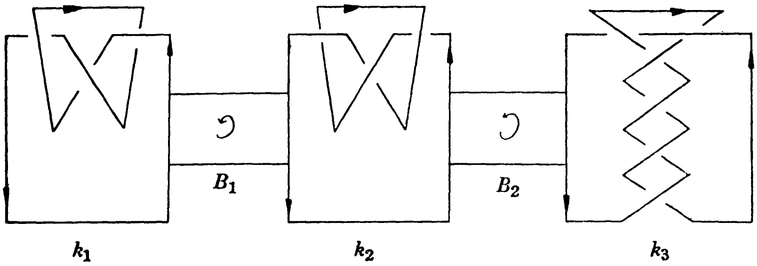

Figure 2.

will be pairwise disjoint. Then the union

$$D = v_1k_1 \cup B_1 \cup v_2k_2 \cup B_2 \cup \cdots \cup B_{n-1} \cup v_nk_n$$

is a 2-cell which lies in the lower half-space bounded by J, and which has just n singular points $v_1, \cdots, v_n$, with singularity types $\kappa_1, \cdots, \kappa_n$ respectively. The boundary of D is a knot $k \subset J$ representing the knot type $\kappa_1 + \cdots + \kappa_n$.

By hypothesis, k is a slice knot. Hence there exists a non-singular 2-cell $D' \subset R^4$ which lies above the hyperplane J, and which spans k. That is:

$$\partial D' = D' \cap J = k .$$

The union

$$m = D \cup D'$$

is now the required 2-sphere.

To summarize, we have proved that *a collection* $\{\kappa_1, \cdots, \kappa_n\}$ *of knot types can occur as the collection of singularities of a 2-sphere in 4-space if and only if* $\kappa_1 + \cdots + \kappa_n$ *is the type of a slice knot.*

Here is another chracterization of slice knots. Let us call the singularity of m at x removable if there exists a modified 2-sphere m' which coincides with m except within an arbitrary small neighborhood U of x, and such that m' has no singularities within U.

Lemma 1. *The singularity at x is removable if and only if it is a slice type.*

Proof. Let N be the star neighborhood of x. If the singularity is removable, then the knot $m \cap \partial N \subset \partial N$ spans a non-singular 2-disk $m' \cap N$ within the 4-cell N. Hence it is a slice knot. Conversely if $m \cap \partial N$ spans a non-singular 2-disk $D \subset N$ then the 2-sphere

$$m' = (m - N) \cup D$$

will have no singularities within N (even on the boundary!). In order to replace

N by a smaller neighborhood, it is only necessary to subdivide before performing this construction. This completes the proof.

2. The polynomial condition

Theorem 2. *If κ is a slice type, its Alexander polynomial is of the form*[5] $A(t) \doteq p(t)p(1/t)$, *where $p(t)$ is a polynomial with integral coefficients.*

Proof. Let m be a locally flat 2-sphere in the 4-space $R^4 \subset S^4$ and let J be a hyperplane of R^4 such that the knot $k = m \cap J$ is of type κ in J. Let H be one of the (closed) half-spaces into which R^4 is separated by J. A tubular neighborhood V of the 2-cell $D = m \cap H$ in H is[6] of the form $D \times C$, where C denotes a 2-cell, and $V \cap J$ is just $k \times C$. Consider the closure Q of $H - V$ in the sphere S^4, and note that the boundary ∂Q of Q is the union of $D \times \partial C$ and the closure W of $J - V = J - (k \times C)$ matched along the torus $k \times \partial C$. It is easy to check that the 1-dimensional homology groups of ∂Q and Q are both infinite cyclic and that an isomorphism between them is induced by the inclusion $\partial Q \subset Q$. Let $\tilde{Q}$ denote the infinite cyclic covering of Q. According to Milnor [9, Lemma 4] there is a "torsion invariant" $\Delta(\tilde{Q})$ associated with this covering. This invariant is a rational function $\Delta(\tilde{Q}) = a(t)/b(t)$ where $a(t)$ and $b(t)$ are non-zero polynomials with integral coefficients; it is well-defined up to sign and multiplication by powers of t.

The corresponding infinite cyclic covering of ∂Q is $\partial\tilde{Q}$. According to Milnor [9, Theorem 2] the torsion invariant $\Delta(\partial\tilde{Q})$ is also defined, and given by the formula

$$\Delta(\partial\tilde{Q}) \doteq \Delta(\tilde{Q})\bar{\Delta}(\tilde{Q}),$$

where the bar indicates the operation $t \to 1/t$ of conjugation.

We can also compute $\Delta(\partial\tilde{Q})$ directly by referring to the subcomplex W and its infinite cyclic covering $\tilde{W}$. According to Milnor [9, Theorem 4] the invariant $\Delta(\tilde{W})$ is defined, and

$$\Delta(\tilde{W}) \doteq A(t)/(t-1),$$

where $A(t)$ denotes the Alexander polynomial of the knot $k \subset J$. Similarly, there is defined a relative torsion invariant $\Delta(\partial\tilde{Q}, \tilde{W})$, and

$$\Delta(\partial\tilde{Q}) \doteq \Delta(\tilde{W})\Delta(\partial\tilde{Q}, \tilde{W}).$$

Note that the pair $(\partial Q, W)$ can be reduced by excision to the pair $(D \times \partial C, k \times \partial C)$. Straightforward computation shows that

5) The notation $A_1(t) \doteq A_2(t)$ means $A_1(t) = \pm t^n A_2(t)$ for some integer n.
6) cf. [11].

$$\Delta(\partial\tilde{Q}, \tilde{W}) \doteq \Delta(D \times \partial\tilde{C}, k \times \partial\tilde{C}) = 1/(t-1)\,.$$

Hence

$$\Delta(\tilde{Q})\bar{\Delta}(\tilde{Q}) \doteq A(t)/(t-1)^2\,,$$

so that

$$A(t) \doteq c(t)c(1/t)\,,$$

where $c(t)$ denotes the rational function $(t-1)\Delta(\tilde{Q})$. Since the ring of integral L-polynomials is a unique factorization domain, $c(t)$ can be expressed as the quotient $a(t)/b(t)$ of two relatively prime polynomials. Let $d(t)$ denote the greatest common divisor of $a(1/t)$ and $b(t)$. Then $a(1/t)=p(1/t)\,d(t)$ and $b(t)=q(t)\,d(t)$, and we have

$$c(t)c(1/t) = p(t)p(1/t)/q(t)q(1/t)\,,$$

where the numerator and denominator are relatively prime. But we know that this quotient is, in fact, a polynomial. Consequently we must have $q(t)\doteq 1$, and so $A(t)\doteq p(t)p(1/t)$ as claimed.

REMARK. Our original proof of Theorem 2 was substantially the same as the proof sketched by H. Terasaka [15]. The proof presented here avoids the rather horrendous calculations of the original.

3. The knot cobordism group

Lemma 3. *If κ is any knot type, then $\kappa+(-\kappa)$ is a slice type.*

Proof. This follows immediately by applying Theorem 1 to the 2-sphere in R^4 which is obtained by "suspending" a representative knot in R^3. Alternatively, here is a direct proof. We will use coordinates x_1, x_2, x_3, x_4 in R^4. Let k be an oriented knot representative of κ that lies above the horizontal plane $x_3=x_4=0$ in the 3-space $x_4=0$. A representative k' of $-\kappa$ may be obtained by reversing the orientation of k and reflecting it in 3-space about this plane. Thus we see that in the 3-space $x_4=0$ there is a representative k'' of $\kappa+(-\kappa)$ that is symmetric about the horizontal plane $x_3=x_4=0$ and intersects it in just two points. Then the set of points (x_1, x_2, x_3, x_4) of 4-space such that $(x_1, x_2, |x_3|+|x_4|)\in k''$ forms a locally flat 2-sphere whose intersection with the hyperplane $x_4=0$ is just k''. (In Figure 3 some cross-sections of this 2-sphere by hyperplanes parallel to $x_4=0$ are shown for the case of the trefoil knot $\kappa=3_1$.)

Lemma 3′. *If κ_1 and κ_2 are both slice types then so is $\kappa_1+\kappa_2$.*

Proof. Given spheres m_1 and m_2 with κ_1 and κ_2 as their respective only singularities, it is easy to construct a sphere m with κ_1 and κ_2 as its only singularities. The lemma therefore follows from Theorem 1.

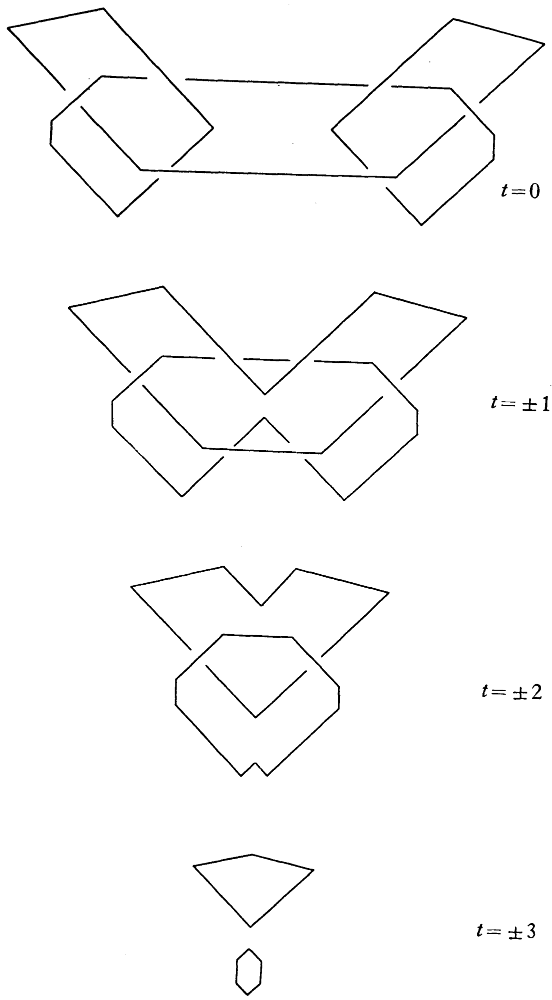

Figure 3.

Lemma 3″. *If λ and $\kappa+\lambda$ are both slice types then so is κ.*

Proof. By Theorem 1′ there is a 2-sphere m in 4-space R^4 that has only two singularities: κ at a point x and λ at a point y. But by Lemma 1 the singularity at y is removable. Hence there exists a 2-sphere m' whose only singularity is κ at x. This proves 3″.

Now let us write $\kappa\sim\lambda$ to mean that $\kappa+(-\lambda)$ is a slice type, and let us check that $\sim$ is an equivalence relation. By Lemma 3 we have $\kappa\sim\kappa$. If $\kappa+(-\lambda)$ is a slice type then so is $-(\kappa+(-\lambda))=\lambda+(-\kappa)$; hence $\kappa\sim\lambda$ implies $\lambda\sim\kappa$. If $\kappa+(-\lambda)$ and $\lambda+(-\mu)$ are slice types then $(\kappa+(-\lambda))+(\lambda+(-\mu)=$

$(\kappa+(-\mu))+(\lambda+(-\lambda))$ is a slice type by Lemma 3′. Since $\lambda+(-\lambda)$ is a slice type according to Lemma 3, it follows from Lemma 3″ that $\kappa+(-\mu)$ must be a slice type. Thus $\kappa\sim\lambda$ and $\lambda\sim\mu$ implies $\kappa\sim\mu$.

Let us write $\langle\kappa\rangle$ for the equivalence class determined by the type κ. It follows easily from Lemma 3′ that the sum operation

$$\langle\kappa\rangle+\langle\lambda\rangle=\langle\kappa+\lambda\rangle$$

is well defined. Thus the set $\mathcal{G}$ of equivalence classes inherits the operation+ from the semigroup $\mathcal{A}$, and with respect to this operation forms an abelian group. The identity element of this group is the class $\langle 0\rangle$ of slice knots, and the inverse of a class $\langle\kappa\rangle$ is the class $-\langle\kappa\rangle=\langle-\kappa\rangle$.

Theorem 3. *In order that $\kappa_0\sim\kappa_1$ it is necessary and sufficient that there exist in the 4-dimensional slab $0\leq x_4\leq 1$ of R^4 a locally flat annulus A whose boundaries are knots k_0 in the hyperplane $x_4=0$ and k_1 in the hyperplane $x_4=1$ representing the types κ_0, κ_1 respectively, the orientations being such that k_0 is homologous to k_1 within A.*

Proof. If such an annulus A exists, then choosing a vertex v below the hyperplane $x_4=0$ and choosing a vertex w above the hyperplane $x_4=1$, the cones vk_0 and wk_1 will be disjoint from each other and from the interior of A. The union

$$m=vk_0\cup A\cup wk_1$$

is then a 2-sphere with just two singularities: κ_0 at v and $-\kappa_1$ at w.

Conversely, given a 2-sphere with just two singularities, it is not diffiicult to move it until it intersects the slab $0\leq x_4\leq 1$ in a non-singular annulus whose boundary curves represent the appropriate knot types.

In view of this theorem we may call the equivalence relation $\sim$ *cobordism*, and the group $\mathcal{G}$ the *knot cobordism group*. (Similar cobordism groups for higher dimensional differentiable knots have been studied by A. Haefliger, M. Kervaire and J. Levine. See for example [6].)

Since there are knot types (many of them) that do not satisfy the polynomial condition of §2, the group $\mathcal{G}$ is non-trivial. Actually $\mathcal{G}$ is not even finitely generated; this can be seen, for example, by observing that there are an infinite number of knots of genus 1, whose polynomials are quadratic, irreducible and distinct from one another.

Murasugi [10] has shown that the signature of the quadratic form associated with a knot is a cobordism invariant[7]. This implies in particular that the clover leaf knot 3_1 determines an element of infinite order in $\mathcal{G}$. It is not known

7) This is strikingly reminiscent of the situation in the classical Thom cobordism theory.

whether or not the quotient group

$$\mathcal{G}/(\text{elements of finite order})$$

is finitely generated.

Any invertible, amphicheiral[8] knot that is not a slice knot determines in $\mathcal{G}$ an element of order 2. An example is provided by the figure eight knot 4_1. However it is not known whether or not $\mathcal{G}$ has any elements of order>2. Neither is it known whether an element of order 2 is necessarily determined by an amphicheiral knot.

An analogous concept of cobordism between links can also be studied [4, 10]. Among the cobordism invariants of a link are the higher order linking numbers $\mu(i_1,\cdots,i_r)$ of reference [8]. (Unpublished.)

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8) Compare [2, pp. 8–11] or [16].

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