FUSION RULES AMONG θ-TWISTED MODULES FOR LATTICE VERTEX OPERATOR ALGEBRAS V_L

Danquynh NGUYEN

Citation	Osaka Journal of Mathematics. 59(1); 217-234
Issue Date	2022-01
Textversion	Publisher
Right	©Departments of Mathematics of Osaka University and Osaka City University.
DOI	10.18910/86341
Is Identical to	https://doi.org/10.18910/86341
Relation	The OJM has been digitized through Project Euclid platform http://projecteuclid.org/ojm starting from Vol. 1, No. 1.

Placed on : Osaka City University Repository

Osaka City University

Nguyen, D.
Osaka J. Math.
59 (2022), 217–234

FUSION RULES AMONG θ-TWISTED MODULES FOR LATTICE VERTEX OPERATOR ALGEBRAS V_L

Danquynh NGUYEN

(Received March 19, 2019, revised December 3, 2020)

Abstract

Let L be a positive-definite, even, integral lattice and θ an automorphism of a central extension of L. In this paper, we study the lattice vertex operator algebra V_L associated with L and its θ-twisted modules. We also discuss the fusion products of V_L-modules and completely determine the fusion rules among them.

1. Introduction

The theory of vertex operator algebras is relatively new compared to other branches of mathematics and has evolved quite rapidly since its inception in the late 1980s. Motivated by the representation theory of affine Lie algebras and the "moonshine module", Borcherds introduced the mathematical formulation of *vertex algebras* in 1986 [4]. Two years later, Frenkel, Lepowsky, and Meurman modified Borcherds's definition and introduced *vertex operator algebras* in their foundational work [14] on the subject. An active field of mathematical research took off from there. The theory of vertex operator algebras was motivated by and has applications in many areas of mathematics, such as number theory, group theory, the theory of modular functions, etc. Vertex (operator) algebras are the mathematical local counterpart of what theoretical physicists call "chiral algebras" in two-dimensional conformal field theory.

In his original paper [4], Borcherds developed a new abstract theory of what he called *vertex operators* by using the explicit structure of an even integral lattice L. Specifically, for any such lattice, he constructed a space on which the vertex operators corresponding to the elements in $\mathbb{C}[L]$ act. These actions were shown to satisfy infinitely many relations, which then formed the axioms in the definition of a vertex algebra. In other words, the vertex algebra of an even lattice is the original example of vertex algebras.

In this paper, we study the lattice vertex operator algebra V_L associated with a positive-definite even lattice and completely determine its fusion rules. For a vertex operator algebra V with irreducible modules M^1, M^2, and M^3, the *fusion rule* of type $\binom{M^3}{M^1\ M^2}$ is defined to be the dimension of the vector space formed by all *intertwining operators* of this type. In conformal field theory, these numbers are closely related to the *fusion coefficients* N_{ij}^k in the operator product expansion of two conformal families $[\phi_i]$ and $[\phi_j]$:

$$[\phi_i] \times [\phi_j] = \sum_k N_{ij}^k [\phi_k]$$

2020 Mathematics Subject Classification. Primary 17B69; Secondary 17B67.

(see [5]). Roughly speaking, the fusion coefficients N_{ij}^k give the scattering amplitudes of the outgoing primary fields ϕ_k when two primary fields ϕ_i and ϕ_j come into contact. We shall see that the above equation is exactly the physical counterpart of what is called a *fusion product* in mathematics literature.

Let us now give an overview of this paper. Let L be a positive-definite, even, integral lattice of rank d and denote by L° its dual lattice. Since L is even, one can show that $L \subseteq L^\circ$. We set $S = \{\lambda_1, \dots, \lambda_k\}$ to be the complete set of representatives of equivalence classes of L in L°. It is well known that $\{V_{L+\lambda} \,|\, \lambda \in S\}$ is the complete list of (inequivalent) irreducible (*untwisted*) V_L-modules (see [14] and [6]). There are also V_L-modules of *twisted* type, whose construction is outlined as follows.

First, denote by $\hat{L}$ be the central extension of L by the cyclic group

$$\mathbb{Z}_2 = \left\langle \kappa \,|\, \kappa^2 = 1 \right\rangle = \langle -1 \rangle .$$

Let $\theta \in \mathrm{Aut}(\hat{L})$ be an automorphism of $\hat{L}$ such that $\theta^2 = \mathrm{id}_{\hat{L}}$ and $\theta(\kappa) = -\kappa$. Let T_χ be the irreducible $\hat{L}/K$-module, where $K = \{a^{-1}\theta(a) \,|\, a \in \hat{L}\}$, associated to a central character $\chi : Z(\hat{L}/K) \to \mathbb{C}^\times$ which sends $\kappa K = (-1)K$ to -1; that is, T_χ is an irreducible $\hat{L}/K$-module on which $\kappa K = (-1)K$ acts as -1. Finally, set $V_L^{T_\chi} = M(1)(\theta) \otimes T_\chi$, then the set of all $V_L^{T_\chi}$ such that T_χ is an irreducible $\hat{L}/K$-module associated to central character χ is the complete list of irreducible V_L-modules of twisted type (see [7]).

For any vertex operator algebra V, the fusion product of two irreducible V-modules M^1 and M^2 is defined by a universal property. The pair $(M, \mathcal{Y})$ is called the **fusion product** of M^1 and M^2 if M is a V-module and $\mathcal{Y}$ is an intertwining operator of type $\binom{M}{M^1\, M^2}$ such that for any V-module W and any intertwining operator $\mathcal{Y}_W$ of type $\binom{W}{M^1\, M^2}$, there exists a unique V-module homomorphism $f : M \to W$ such that $\mathcal{Y}_W = f \circ \mathcal{Y}$. The fusion product of M^1 and M^2 is denoted by $M^1 \boxtimes_V M^2$. If V is a rational, C_2-cofinite vertex operator algebra, then the fusion product of any two irreducible V-modules always exists [17], in which case we use the following definition:

$$M^1 \boxtimes_V M^2 := \sum_i N_V\binom{M^i}{M^1 M^2} M^i \tag{1}$$

where M^i runs over the set of equivalence classes of irreducible V-modules and the symbol $N_V\binom{M^i}{M^1 M^2}$ denotes the dimension of the space formed by all intertwining operators of type $\binom{M^i}{M^1 M^2}$, namely, the fusion rule of type $\binom{M^i}{M^1 M^2}$. We note that this definition works for when M^i are irreducible θ-twisted modules since the category of θ-twisted modules is semi-simple and thus a fusion product of two irreducible θ-twisted modules is defined.

Our main object of interest, the lattice VOA V_L, is known to be rational and C_2-cofinite, and thus the fusion products of its modules always exist. The fusion product of two untwisted irreducible V_L-modules is well-known, namely $V_{L+\lambda} \boxtimes_{V_L} V_{L+\mu} = V_{L+\lambda+\mu}$ (see [9], Proposition 12.9). In this paper, we determine the other two fusion products: $V_{L+\lambda} \boxtimes_{V_L} V_L^{T_\chi}$ and $V_L^{T_{\chi_1}} \boxtimes_{V_L} V_L^{T_{\chi_2}}$ by a method briefly outlined here. We invoke a result proved in [2], which says that the fusion rule of type $\binom{M^1}{M^2\, M^3}$ for V_L is either 0 or 1 for any irreducible module M^i for V_L. For $V_{L+\lambda} \boxtimes_{V_L} V_L^{T_\chi}$, we show that it is equal to $V_L^{T_{\chi(\lambda)}}$ (a twisted V_L-module deter-

mined by λ and χ) by showing that the fusion rule $N_{V_L}\binom{V_L^{T_{\chi(\lambda)}}}{V_{L+\lambda}\ V_L^{T_\chi}} = 1$ and all other fusion rules $N_{V_L}\binom{M}{V_{L+\lambda}\ V_L^{T_\chi}} = 0$ where M is any other irreducible V_L-module. This is proved by an explicit construction of a non-trivial intertwining operator of type $\binom{V_L^{T_{\chi(\lambda)}}}{V_{L+\lambda}\ V_L^{T_\chi}}$. In almost exactly the same way, we can determine the fusion product $V_L^{T_{\chi_1}} \boxtimes_{V_L} V_L^{T_{\chi_2}}$.

This paper is organized as follows. In Section 2, we recall the definitions and some important results about intertwining operators and fusion rules. Section 3 contains the construction of vertex operator algebra V_L and its modules. Section 4 reviews a well-known result by Dong and Lepowsky [9] concerning the fusion product of two untwisted V_L-modules, namely $V_{L+\lambda} \boxtimes_{V_L} V_{L+\mu}$. The last two sections are heart of this paper, where we give detailed computations of the two fusion products: $V_{L+\lambda} \boxtimes_{V_L} V_L^{T_\chi}$ and $V_L^{T_{\chi_1}} \boxtimes_{V_L} V_L^{T_{\chi_2}}$.

2. Intertwining operators and fusion rules

Throughout this paper, we denote by V a vertex operator algebra (over the complex number field) with vacuum vector $\mathbf{1}$ and conformal vector ω.

DEFINITION 2.1. Let (M^i, Y_{M^i}) $(i = 1, 2, 3)$ be weak V-modules. An **intertwining operator** of type $\binom{M^3}{M^1\ M^2}$ is a linear map:

$$\mathcal{Y} = \mathcal{Y}(\cdot, z) : M^1 \to (\mathrm{Hom}(M^2, M^3))\{z\}$$
$$u \mapsto \mathcal{Y}(u, z) = \sum_{n\in\mathbb{C}} u_n z^{-n-1}, \quad \text{where } u_n \in \mathrm{Hom}(M^2, M^3)$$

satisfying the following properties:

(1) For any $u \in M^1, v \in M^2$, and $\lambda \in \mathbb{C}$, $u_{m+\lambda}v = 0$ for sufficiently large integer m,

(2) For any $a \in V, u \in M^1$, the Jacobi identity holds:

$$z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right) Y_{M^3}(a, z_1)\mathcal{Y}(u, z_2) - z_0^{-1}\delta\left(\frac{z_2 - z_1}{-z_0}\right)\mathcal{Y}(u, z_2)Y_{M^2}(a, z_1)$$
$$= z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right)\mathcal{Y}(Y_{M^1}(a, z_0)u, z_2),$$

(3) For $u \in M^1$, the $L(-1)$ derivative property is satisfied:

$$\mathcal{Y}(L(-1)u, z) = \frac{d}{dz}\mathcal{Y}(u, z).$$

DEFINITION 2.2. Let g_1, g_2, g_3 be automorphisms of V and let M^1, M^2, and M^3 be g_1-, g_2-, and g_3-twisted V-modules, respectively. A **twisted intertwining operator** of type $\binom{M^3}{M^1\ M^2}$ is a linear map:

$$\mathcal{Y} = \mathcal{Y}(\cdot, z) : M^1 \to (\mathrm{Hom}(M^2, M^3))\{z\}[\log z]$$
$$u \mapsto \mathcal{Y}(u, z) = \sum_{k=0}^{K}\sum_{n\in\mathbb{C}} u_{n,k} z^{-n-1}(\log z)^k, \quad \text{where } u_n \in \mathrm{Hom}(M^2, M^3)$$

satisfying the lower truncation property, the duality property, and the $L(-1)$ derivative property as detailed in [16].

Denoting by $\mathcal{I}_V\binom{M^3}{M^1\ M^2}$ the vector space spanned by all intertwining operators of type $\binom{M^3}{M^1\ M^2}$, we have the following definition.

DEFINITION 2.3. The **fusion rule** of type $\binom{M^3}{M^1\ M^2}$ for V is defined by

$$N_V\binom{M^3}{M^1\ M^2} = \dim \mathcal{I}_V\binom{M^3}{M^1\ M^2}.$$

Fusion rules have the following well-known symmetries (see [13], Propositions 5.4.7 and 5.5.2):

Proposition 2.4. *Let M_i $(i = 1, 2, 3)$ be V-modules and M_i' the corresponding contragredient modules, then*

$$N_V\binom{M_3}{M_1\ M_2} = N_V\binom{M_3}{M_2\ M_1} = N_V\binom{M_2'}{M_1\ M_3'}.$$

We also quote here a useful result from [3], which shall be invoked repeatedly in the derivation of our main results:

Proposition 2.5. *Let V be a vertex operator algebra and let M^1, M^2, M^3 be V-modules, where M^1 and M^2 are irreducible. Suppose that U is a vertex operator subalgebra of V (with the same Virasoro element) and that N^1 and N^2 are irreducible U-submodules of M^1 and M^2, respectively. Then the restriction map from $\mathcal{I}_V\binom{M^3}{M^1\ M^2}$ to $\mathcal{I}_U\binom{M^3}{N^1\ N^2}$ is injective. In particular,*

$$\dim \mathcal{I}_V\binom{M^3}{M^1\ M^2} \leq \dim \mathcal{I}_U\binom{M^3}{N^1\ N^2}.$$

DEFINITION 2.6. Let V be a vertex operator algebra and M^1, M^2 its modules. The **fusion product** of M^1 and M^2 is a V-module, denoted by $M^1 \boxtimes_V M^2$, together with an intertwining operator $\mathcal{Y} \in \mathcal{I}_V\binom{M^1 \boxtimes_V M^2}{M^1\ M^2}$ that satisfies the following universal property: For any V-module W and $\mathcal{Y}_W \in \mathcal{I}_V\binom{W}{M^1\ M^2}$, there exists a unique V-module homomorphism $f : M^1 \boxtimes_V M^2 \to W$ such that $\mathcal{Y}_W = f \circ \mathcal{Y}$.

REMARK. A fusion product may not exist; but when it does, it is unique up to isomorphism as a consequence of the universal property.

If V is a rational and C_2-cofinite vertex operator algebra, then the fusion product of any two irreducible V-modules exists (Proposition 4.13 in [17]). Motivated by the concept of a *fusion algebra* in conformal field theory (Equation (2.130) in [5]), we define the fusion product, if it exists, as follows:

$$M^1 \boxtimes_V M^2 = \sum_i N_V\binom{M^i}{M^1 M^2} M^i,$$

where M^i runs over the set of equivalence classes of irreducible V-modules. If the context is clear, we may drop the subscript V in $M^1 \boxtimes_V M^2$ and simply write $M^1 \boxtimes M^2$.

3. The vertex operator algebra V_L and its modules

Let L denote a positive-definite even lattice of rank d, that is, L is a free abelian group of rank d equipped with a $\mathbb{Z}$-valued non-degenerate, positive-definite symmetric $\mathbb{Z}$-bilinear form $\langle\ ,\ \rangle : L \times L \to \mathbb{Z}$ satisfying $\langle \alpha, \alpha \rangle \in 2\mathbb{Z}$ for $\alpha \in L$. The form being non-degenerate means that if $\langle \alpha, L \rangle = \{0\}$, then $\alpha = 0$, while being positive-definite means $\langle \alpha, \alpha \rangle > 0$ for any non-zero $\alpha \in L$. Our main interests are V_L, whatever this symbol means at this point, and its irreducible modules. The space V_L is a tensor product of $M(1)$ and $\mathbb{C}[L]$; therefore, we first recall a construction of $M(1)$.

3.1. The vertex operator algebra $M(1)$ and its modules. Let $\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Z}} L$ be the complex extension of L, then $\mathfrak{h}$ is a d-dimensional vector space which naturally inherits a $\mathbb{C}$- bilinear form $\langle\ ,\ \rangle$ as the extension of the form on L. The lattice L is identified with $1 \otimes_{\mathbb{Z}} L$ as a $\mathbb{Z}$-submodule of $\mathfrak{h}$. Viewing $\mathfrak{h}$ as an abelian Lie algebra, we define the following Lie algebra affinization:

$$\hat{\mathfrak{h}} = \mathfrak{h} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}C$$

with the following commutation relations:

$$[\alpha_1 \otimes t^m, \alpha_2 \otimes t^n] = m \langle \alpha_1, \alpha_2 \rangle \delta_{m+n,0} C, \qquad [C, \hat{\mathfrak{h}}] = 0$$

for any $\alpha_1, \alpha_2 \in \mathfrak{h}$ and any $m, n \in \mathbb{Z}$. The Lie algebra $\hat{\mathfrak{h}}$ has an abelian Lie subalgebra

$$\hat{\mathfrak{h}}^+ = \mathfrak{h} \otimes \mathbb{C}[t] \oplus \mathbb{C}C.$$

For any $\lambda \in \mathfrak{h}$, let $\mathbb{C}e^\lambda$ denote the 1-dimensional $\hat{\mathfrak{h}}^+$-module with module actions defined by

$$h \otimes t\mathbb{C}[t] \cdot e^\lambda = \{0\}, \quad h \otimes t^0 \cdot e^\lambda = \langle \lambda, h \rangle e^\lambda, \quad C \cdot e^\lambda = e^\lambda$$

for $h \in \mathfrak{h}$. Now consider the induced $\hat{\mathfrak{h}}$-module:

$$M(1, \lambda) = \mathrm{Ind}_{\hat{\mathfrak{h}}^+}^{\hat{\mathfrak{h}}} \mathbb{C}e^\lambda = U(\hat{\mathfrak{h}}) \otimes_{U(\hat{\mathfrak{h}}^+)} \mathbb{C}e^\lambda \cong S(t^{-1}\mathbb{C}[t^{-1}]) \otimes \mathfrak{h},$$

where $U(\cdot)$ denotes the universal enveloping algebra and $S(\cdot)$ the symmetric algebra. The action of $h \otimes t^n \in \hat{\mathfrak{h}}$ on any $\hat{\mathfrak{h}}$-module is denoted by $h(n)$ ($h \in \mathfrak{h}$, $n \in \mathbb{Z}$). The space $M(1, 0)$ is spanned by vectors of the form $v = h_1(-n_1) \cdots h_k(-n_k) \otimes e^0$ where $h_i \in \mathfrak{h}$ and $n_i \geq 1$. The vertex operator of v on $M(1, \lambda)$ is given by the following linear map

$$Y = Y(\cdot, z) : M(1, 0) \to (\mathrm{End} M(1, \lambda))[[z, z^{-1}]],$$

$$Y(v, z) := {}^{\circ}_{\circ} \left(\frac{1}{(n_1 - 1)!} \left(\frac{d}{dz} \right)^{n_1 - 1} h_1(z) \right) \cdots \left(\frac{1}{(n_k - 1)!} \left(\frac{d}{dz} \right)^{n_k - 1} h_k(z) \right) {}^{\circ}_{\circ},$$

where $h_i(z) = \sum_{n \in \mathbb{Z}} h_i(n) z^{-n-1}$.

The symbol ${}^{\circ}_{\circ} \cdot {}^{\circ}_{\circ}$ denotes a *normally ordered product* (or *normal ordering*) which rearranges the items enclosed between the colons so that the operators $h_i(n)$, for $n < 0$, are to be placed to the left of the operators $h_i(n)$, for $n > 0$, before the multiplication is performed. When $\lambda = 0$, we simply write $M(1) = M(1, 0)$.

Suppose that $\{\beta_1, \ldots, \beta_d\}$ is an orthonormal basis of $\mathfrak{h}$ $(= \mathbb{C} \otimes_{\mathbb{Z}} L)$ with respect to the form $\langle\ ,\ \rangle$. We use the notations $\mathbf{1}$ and ω to denote the following two elements of $M(1)$:

$$\mathbf{1} = 1 \otimes e^0 \in M(1), \quad \omega = \frac{1}{2} \sum_{i=1}^{d} \beta_i(-1)\beta_i(-1) \otimes e^0 \in M(1).$$

Then, as shown in [14], $(M(1), Y(\cdot, z), \mathbf{1}, \omega)$ is a simple vertex operator algebra and $M(1, \lambda)$, where $\lambda \in \mathfrak{h}$, are the irreducible $M(1)$-modules.

3.2. The lattice vertex operator algebra V_L and its modules. We closely follow the set-up in [14]. Let $(\hat{L}, -)$ be the central extension of L by the cyclic group $\langle\kappa\rangle = \left\langle \kappa \mid \kappa^2 = 1 \right\rangle$. This means that we have the following exact sequence

$$1 \longrightarrow \langle\kappa\rangle = \langle -1 \rangle \longrightarrow \hat{L} \xrightarrow{\ -\ } L \longrightarrow 0$$

associated with a commutator map

$$c : L \times L \to \mathbb{C}^\times, \quad c(\alpha, \beta) = \kappa^{\langle \alpha, \beta \rangle} = (-1)^{\langle \alpha, \beta \rangle}$$

for any $\alpha, \beta \in L$. Let $e : L \to \hat{L}, \alpha \mapsto e_\alpha$ be a section such that $0 \mapsto e_0 = 1$. Then we have $\hat{L} = \{\kappa^i e_\alpha \mid \alpha \in L, i = 0, 1\}$. This section defines a 2-cocycle given by

$$\epsilon : L \times L \to \mathbb{C}^\times, \quad e_\alpha e_\beta = \epsilon(\alpha, \beta) e_{\alpha+\beta}$$

In [14], the following properties of ϵ are known for any $\alpha, \beta, \gamma \in L$

$$\begin{aligned} \epsilon(\alpha, \beta)\epsilon(\alpha + \beta, \gamma) &= \epsilon(\beta, \gamma)\epsilon(\alpha, \beta + \gamma), \\ \epsilon(\alpha, \beta)(\epsilon(\beta, \alpha))^{-1} &= c(\alpha, \beta), \\ \epsilon(\alpha, 0) = \epsilon(0, \alpha) &= 1. \end{aligned}$$

We next discuss the group algebra $\mathbb{C}[L] = \bigoplus_{\lambda \in L} \mathbb{C} e^\lambda$, which is an $\hat{L}$-module under the actions

$$\hat{L} \times \mathbb{C}[L] \to \mathbb{C}[L], \quad e_\alpha \cdot e^\lambda = \epsilon(\alpha, \lambda) e^{\alpha+\lambda}, \quad \kappa \cdot e^\lambda = -e^\lambda$$

for any $\alpha, \lambda \in L$. We are now ready to define

$$V_L = M(1) \otimes \mathbb{C}[L].$$

The $\hat{\mathfrak{h}}$-module structure of $M(1)$ extends naturally to the $\hat{\mathfrak{h}}$-module structure of V_L

$$\hat{\mathfrak{h}} \times V_L \to V_L$$

$$h(n) \cdot (u \otimes e^\lambda) = (h(n) \cdot u) \otimes e^\lambda \ (n \neq 0), \quad h(0) \cdot (u \otimes e^\lambda) = \langle h, \lambda \rangle (u \otimes e^\lambda), \quad C \cdot (u \otimes e^\lambda) = u \otimes e^\lambda$$

for any $h \in \mathfrak{h}, u \in M(1)$, and $\lambda \in L$.

Next, we explain that V_L has the structure of a vertex operator algebra. For each $v \in V_L, v = h_1(-n_1) \cdots h_k(-n_k) \otimes e^\lambda$ for $\lambda \in L$, $h_i \in \mathfrak{h}$, and $n_i \geq 1$. We define the vertex operator associated to e^λ by

$$Y(e^\lambda, z) = \exp\left(\sum_{n=1}^{\infty} \frac{\lambda(-n)}{n} z^n \right) \exp\left(- \sum_{n=1}^{\infty} \frac{\lambda(n)}{n} z^{-n} \right) e_\lambda z^\lambda.$$

Note that $\mathbb{C}[L]$ is an $\hat{L}$-module as described above, so e_λ is the left action of $e_\lambda \in \hat{L}$ on $\mathbb{C}[L]$. The operator z^λ on $\mathbb{C}[L]$ is defined by

$$z^\lambda \cdot e^\mu = z^{\langle \lambda, \mu \rangle} e^\mu.$$

We then define the vertex operator associated to $v \in V_L$ by

$$Y = Y(\cdot, z) : V_L \to (\mathrm{End} V_L)\{z\},$$

$$Y(v,z) = {}^{\circ}_{\circ} \left(\frac{1}{(n_1-1)!} \left(\frac{d}{dz} \right)^{n_1-1} h_1(z) \right) \cdots \left(\frac{1}{(n_k-1)!} \left(\frac{d}{dz} \right)^{n_k-1} h_k(z) \right) Y(e^\lambda, z) {}^{\circ}_{\circ}.$$

With $\mathbf{1} = 1 \otimes e^0 \in M(1) \subseteq V_L$ and $\omega = \frac{1}{2} \sum_{i=1}^{d} \beta_i(-1)\beta_i(-1) \otimes e^0 \in M(1) \subseteq V_L$, the quadruple $(V_L, Y, \mathbf{1}, \omega)$ was shown (in [14] and [20]) to be a simple vertex operator algebra.

To classify V_L-modules, we first need to introduce the dual lattice of L, which is denoted by $L^\circ = \{\beta \in \mathfrak{h} | \, \langle \alpha, \beta \rangle \in \mathbb{Z}, \alpha \in L\}$. Since L is an even lattice, one can show that $L \subseteq L^\circ$. Let $S = \{\lambda_1, \ldots, \lambda_k\}$ be the complete set of representatives of equivalence classes of L in its dual lattice L°. Then it follows that

$$\mathbb{C}[L^\circ] = \mathbb{C}[L + \lambda_1] \oplus \cdots \oplus \mathbb{C}[L + \lambda_k],$$
$$V_{L^\circ} = V_{L+\lambda_1} \oplus \cdots \oplus V_{L+\lambda_k},$$

where $V_{L+\lambda_i} = M(1) \otimes \mathbb{C}[L + \lambda_i]$ $(i = 1, 2, \ldots, k)$. It was shown in [14] and [6] that $\{V_{L+\lambda} \, | \, \lambda \in S\}$ is the complete list of (inequivalent) irreducible (*untwisted*) V_L-modules.

The classification of irreducible θ-twisted modules for V_L was done in [7] and is recalled next. Let $\theta \in \mathrm{Aut}(\hat{L})$ be an automorphism of $\hat{L}$ such that $\theta^2 = \mathrm{id}_{\hat{L}}$ and $\theta(\kappa) = -\kappa$. Recall that $\hat{L} = \langle e_\alpha, -e_\alpha \, | \, \alpha \in L \rangle$, so the action of θ on $\hat{L}$ can be viewed as

$$\theta(\kappa^i e_\alpha) = \kappa^i e_{-\alpha}.$$

It can be easily observed that θ induces an automorphism $\bar{\theta}$ on L such that $\bar{\theta}^2 = \mathrm{id}_L$ and $\bar{\theta}(\alpha) = -\alpha$, for any $\alpha \in L$. One can now define the action of θ on V_L by

$$h_1(-n_1) \cdots h_k(-n_k) \otimes e^\alpha \mapsto (-1)^k h_1(-n_1) \cdots h_k(-n_k) \otimes e^{-\alpha},$$

for $h_i \in \mathfrak{h}, n_i \geq 1$, and $\alpha \in L$. In fact, the map θ turns out to be an automorphism of V_L which has two eigensubspaces $V_L^+ = \{v \in V_L \, | \, \theta(v) = v\}$ and $V_L^- = \{v \in V_L \, | \, \theta(v) = -v\}$. A thorough treatment of the fusion rules for V_L^+ has been done in [3], which lays the foundation for our study in this paper.

We now recall a θ-twisted affine Lie algebra $\hat{\mathfrak{h}}[\theta] = \mathfrak{h} \otimes t^{1/2}\mathbb{C}[t, t^{-1}] \oplus \mathbb{C}C$ with the following brackets

$$[\alpha_1 \otimes t^m, \alpha_2 \otimes t^n] = m \langle \alpha_1, \alpha_2 \rangle \delta_{m+n,0} C, \quad [C, \hat{\mathfrak{h}}[\theta]] = 0$$

for all $\alpha_1, \alpha_2 \in \mathfrak{h}$ and $m, n \in \mathbb{Z} + \frac{1}{2}$. The Lie algebra $\hat{\mathfrak{h}}[\theta]$ has the subspaces

$$\hat{\mathfrak{h}}[\theta]^+ = \mathfrak{h} \otimes t^{1/2}\mathbb{C}[t], \quad \hat{\mathfrak{h}}[\theta]^- = \mathfrak{h} \otimes t^{-1/2}\mathbb{C}[t^{-1}].$$

Viewing $\mathbb{C}$ as a module for $\hat{\mathfrak{h}}[\theta]^+ \oplus \mathbb{C}C$ on which $\hat{\mathfrak{h}}[\theta]^+$ acts trivially and C acts as a multiplication by 1, we have the induced module

$$M(1)(\theta) = \mathrm{Ind}_{\hat{\mathfrak{h}}[\theta]^+ \oplus \mathbb{C}C}^{\hat{\mathfrak{h}}[\theta]} \mathbb{C} = U(\hat{\mathfrak{h}}[\theta]) \otimes_{U(\hat{\mathfrak{h}}[\theta]^+ \oplus \mathbb{C}C)} \mathbb{C} \cong S(t^{-1/2}\mathbb{C}[t^{-1}] \otimes \mathfrak{h}).$$

Define $K = \{a^{-1}\theta(a) \mid a \in \hat{L}\}$. Let T_χ be the irreducible $\hat{L}/K$-module associated to a central character $\chi : Z(\hat{L}/K) \to \mathbb{C}^\times$ such that $(-1)K \mapsto -1$ (that is, T_χ is an irreducible $\hat{L}/K$-module on which $(-1)K$ acts as -1). For each such T_χ, define a twisted space by

$$V_L^{T_\chi} = M(1)(\theta) \otimes T_\chi.$$

Then $\{V_L^{T_\chi}\}$, where T_χ is an irreducible $\hat{L}/K$-module as described above, are the irreducible θ-twisted V_L modules. The action of θ on $M(1)(\theta)$ extends to an action on $V_L^{T_\chi}$

$$\theta : V_L^{T_\chi} \to V_L^{T_\chi}, h_1(-n_1)\cdots h_k(-n_k) \otimes t \mapsto (-1)^k h_1(-n_1)\cdots h_k(-n_k) \otimes t$$

for $h_i \in \mathfrak{h}, n_i \in \frac{1}{2}+\mathbb{Z}$, and $t \in T_\chi$. As before, we denote by $V_L^{T_\chi,+}$ and $V_L^{T_\chi,-}$ the eigensubspaces of $V_L^{T_\chi}$ of eigenvalues 1 and -1, respectively.

We can now state two results from [3] and [2] on V_L^+:

Proposition 3.2.1 ([3], Theorem 3.4)**.** *Let L be a positive-definite even lattice and let $\{\lambda_i\}$ be a set of representatives of L°/L. Then any irreducible V_L^+-module is isomorphic to one of the irreducible modules $V_L^\pm, V_{\lambda_i+L}$ with $2\lambda_i \notin L, V_{\lambda_i+L}^\pm$ with $2\lambda_i \in L$ or $V_L^{T_\chi,\pm}$ for a central character χ of $\hat{L}/K$ with $\chi(\kappa) = -1$.*

Proposition 3.2.2 ([2], Proposition 3.3)**.** *Let W^1, W^2, and W^3 be irreducible V_L^+-modules. Then the following hold:*
(1) *The fusion rules $N\binom{W^3}{W^1\ W^2}$ is either zero or one.*
(2) *If all W^i ($i = 1, 2, 3$) are of twisted type, then the fusion rule $N\binom{W^3}{W^1\ W^2}$ is zero.*
(3) *If one of W^i ($i = 1, 2, 3$) is of twisted type and the others are of untwisted type, then the fusion rule $N\binom{W^3}{W^1\ W^2}$ is zero.*

The next three sections discuss the three different fusion products of V_L-modules. The first one, Section 4, is a result directly obtained from [9] concerning modules of untwisted type and the fusion product $V_{L+\lambda} \boxtimes V_{L+\mu}$. Sections 5 and 6 discuss the cases when at least one module of twisted type is involved in the fusion product; specifically, we compute $V_{L+\lambda} \boxtimes V_L^{T_\chi}$ and $V_L^{T_{\chi_1}} \boxtimes V_L^{T_{\chi_2}}$, which are new.

4. Fusion products $V_{L+\lambda} \boxtimes V_{L+\mu}$

For the rest of this paper, we drop the subscript V_L in the fusion rule N_{V_L} and fusion product $\boxtimes_{V_L}$ notations and simply write N and $\boxtimes$, respectively. Recall that $S = \{\lambda_1, \ldots, \lambda_k\}$ is a complete set of representatives of equivalence classes of L in its dual lattice L°. The following proposition is an immediate consequence of Proposition 12.9 [9].

Proposition 4.1. *For any $\lambda, \mu \in S$, we have $V_{L+\lambda} \boxtimes V_{L+\mu} = V_{L+\lambda+\mu}$.*

Proof. Let M^i run over the equivalence classes of irreducible V_L-modules. By the definition of a fusion product, we have

$$V_{L+\lambda} \boxtimes V_{L+\mu} = \sum_i N\binom{M^i}{V_{L+\lambda}\ V_{L+\mu}} M^i$$

$$= \sum_{\nu\in S} N\binom{V_{L+\nu}}{V_{L+\lambda}\ V_{L+\mu}} V_{L+\nu} + \sum_{V_L^{T_\chi}} N\binom{V_L^{T_\chi}}{V_{L+\lambda}\ V_{L+\mu}} V_L^{T_\chi},$$

where $V_L^{T_\chi}$ runs over the equivalence classes of irreducible θ-twisted V_L-modules. Now by Proposition 12.9 in [9], we have

$$N\binom{V_{L+\nu}}{V_{L+\lambda}\ V_{L+\mu}} = 1$$

if and only if $\nu + L = \lambda + \mu + L$. Recall that V_L^+ is a vertex operator subalgebra of V_L, and that $\{V_{L+\lambda} \mid \lambda \in S\}$ is the set of all inequivalent irreducible V_L-modules and $\left\{V_L^{T_\chi}\right\}$ the set of all inequivalent irreducible θ-twisted V_L-modules. By Proposition 2.5, we have

$$N_{V_L}\binom{V_L^{T_\chi}}{V_{L+\lambda}\ V_{L+\mu}} \le N_{V_L^+}\binom{V_L^{T_\chi}}{V_{L+\lambda}\ V_{L+\mu}} = 0.$$

The last equality follows from Proposition 3.2.2 (3). Thus, it follows that

$$V_{L+\lambda} \boxtimes V_{L+\mu} = N\binom{V_{L+\lambda+\mu}}{V_{L+\lambda}\ V_{L+\mu}} V_{L+\lambda+\mu} = V_{L+\lambda+\mu}.$$

□

5. Fusion products $V_{L+\lambda} \boxtimes V_L^{T_\chi}$

Let M^k run over the set of all inequivalent irreducible V_L-modules and θ-twisted V_L-modules, then by the definition of fusion product, we have

$$V_{L+\lambda} \boxtimes V_L^{T_\chi} = \sum_k N\binom{M^k}{V_{L+\lambda}\ V_L^{T_\chi}} M^k = \sum_{\mu\in S} N\binom{V_{L+\mu}}{V_{L+\lambda}\ V_L^{T_\chi}} V_{L+\mu} + \sum_{V_L^{T_{\chi_2}}} N\binom{V_L^{T_{\chi_2}}}{V_{L+\lambda}\ V_L^{T_\chi}} V_L^{T_{\chi_2}},$$

where $V_L^{T_{\chi_2}}$ runs over the equivalence classes of irreducible θ-twisted V_L-modules.

Lemma 5.1. *For any $\lambda, \mu \in L^\circ$ and any central character χ of $\hat{L}/K$ such that $\chi(\kappa) = -1$, we have*

$$N\binom{V_{L+\mu}}{V_{L+\lambda}\ V_L^{T_\chi}} = 0.$$

Proof. For any $\mu \in L^\circ$, the space $V_{L+\mu}$ is a V_L-module and thus is also a V_L^+-module. Recall that $V_L^{T_\chi}$ is a twisted irreducible V_L-module while its submodule $V_L^{T_\chi,+}$ is an irreducible V_L^+-module of twisted type by Proposition 3.2.1. □

CASE 1. If $2\lambda \notin L$, then $V_{L+\lambda}$ is an untwisted irreducible V_L^+-module by Proposition 3.2.1. Therefore, by Propositions 2.4 and 3.2.2 (3), we have

$$N_{V_L}\binom{V_{L+\mu}}{V_{L+\lambda}\ V_L^{T_\chi}} \le N_{V_L^+}\binom{V_{L+\mu}}{V_{L+\lambda}\ V_L^{T_\chi,+}} = 0.$$

CASE 2. If $2\lambda \in L$, then $V_{L+\lambda}^\pm$ are (untwisted) irreducible V_L^+-modules by Proposition 3.2.1. It follows that

$$N_{V_L}\begin{pmatrix} V_{L+\mu} \\ V_{L+\lambda} \; V_L^{T_\chi} \end{pmatrix} \leq N_{V_L^+}\begin{pmatrix} V_{L+\mu} \\ V_{L+\lambda}^+ \; V_L^{T_\chi,+} \end{pmatrix} = 0.$$

We now show that there exists an intertwining operator of type $\binom{V_L^{T_{\chi_1}}}{V_{L+\lambda} \; V_L^{T_\chi}}$ for V_L. We point out that χ_1 is, in fact, determined by both χ and λ by a formula to be given below.

Let $\chi : Z(\hat{L}/K) \to \mathbb{C}^\times$ such that $\chi(\kappa) = -1$ be any central character of $\hat{L}/K$ and T_χ the corresponding irreducible $\hat{L}/K$-module under the action $\kappa \cdot v = -v$ for any $v \in T_\chi$. As shown in Section 4, we have $V_L^{T_\chi} = M(1)(\theta) \otimes T_\chi$, which is a θ-twisted V_L-module.

Let $\lambda \in L^\circ$ and define an automorphism σ_λ of $\hat{L}$ by $\sigma_\lambda(a) = \kappa^{\langle \lambda, \bar{a} \rangle} a = (-1)^{\langle \lambda, \bar{a} \rangle} a$. Let $a \in \hat{L}$, then $\sigma_\lambda(\theta(a)) = \kappa^{\langle \lambda, \overline{\theta(a)} \rangle} \theta(a)$, while $\theta(\sigma_\lambda(a)) = \theta(\kappa^{\langle \lambda, \bar{a} \rangle} a) = \kappa^{\langle \lambda, \bar{a} \rangle} \theta(a)$. Therefore, $\sigma_\lambda(\theta(a)) = \theta(\sigma_\lambda(a))$. For any $a^{-1}\theta(a) \in K$, σ_λ sends it back to K since

$$\sigma_\lambda(a^{-1}\theta(a)) = \sigma_\lambda(a^{-1})\sigma_\lambda(\theta(a)) = (\sigma_\lambda(a))^{-1}\theta(\sigma_\lambda(a)) \in K.$$

Thus, the automorphism σ_λ stabilizes K and consequently induces an automorphism on $\hat{L}/K$ such that $\sigma_\lambda(aK) = \sigma_\lambda(a)K = \kappa^{\langle \lambda, \bar{a} \rangle} aK = (-1)^{\langle \lambda, \bar{a} \rangle} aK$ for any $aK \in \hat{L}/K$.

For any $\hat{L}/K$-module T, we denote by $T \circ \sigma_\lambda$ the $\hat{L}/K$-*module twisted by* σ_λ, namely $T \circ \sigma_\lambda = T$ as vector spaces and there is an action of $\hat{L}/K$ on $T \circ \sigma_\lambda$ which is determined by σ_λ as follows

$$\hat{L}/K \quad \times \quad T \circ \sigma_\lambda (= T) \to T \circ \sigma_\lambda (= T), \quad a \cdot t = \sigma_\lambda(a)t.$$

If $T = T_\chi$, we have

$$\hat{L}/K \quad \times \quad T_\chi \circ \sigma_\lambda (= T_\chi) \to T_\chi \circ \sigma_\lambda (= T_\chi), \; \kappa \cdot t = -t, \; a \cdot t = \sigma_\lambda(a)t$$

for any $a \in \hat{L}/K$ and $t \in T_\chi$. Moreover, the module $T_\chi \circ \sigma_\lambda$ is irreducible since T_χ is irreducible. Since the number of central characters of $\hat{L}/K$ which send κ to -1 is finite ([14], Proposition 7.4.8), there exists a unique central character χ_1 of $\hat{L}/K$ such that the corresponding $\hat{L}/K$-module T_{χ_1} satisfies $T_{\chi_1} \cong T_\chi \circ \sigma_\lambda$. To emphasize the fact that χ_1 is dependent on χ and λ, we use the notation $\chi^{(\lambda)}$ instead of χ_1 and thus have $T_{\chi^{(\lambda)}} \cong T_\chi \circ \sigma_\lambda$. Let f denote this isomorphism: $f : T_\chi \circ \sigma_\lambda \to T_{\chi^{(\lambda)}}$.

Let $\lambda \in L^\circ$ and $\alpha \in L$. Define a linear isomorphism

$$\eta_{\lambda+\alpha} : T_\chi \circ \sigma_\lambda \to T_{\chi^{(\lambda)}}, \quad \eta_{\lambda+\alpha} = \epsilon(-\alpha, \lambda) e_\alpha \circ f$$

Recall that e_α is the left action of $e_\alpha \in \hat{L}$ on $\mathbb{C}[L]$ with the following properties.

Lemma 5.2. *For any $\alpha, \beta \in L$, we have $e_\alpha e_\beta = (-1)^{\langle \alpha, \beta \rangle} e_\beta e_\alpha$ as operators on $\mathbb{C}[L]$.*

Proof. Let $e^\mu \in \mathbb{C}[L]$ for $\mu \in L$. Then it follows that

$$e_\alpha e_\beta \cdot e^\mu = e_\alpha(\epsilon(\beta, \mu) e^{\beta+\mu}) = \epsilon(\beta, \mu)\epsilon(\alpha, \beta + \mu) e^{\alpha+(\beta+\mu)} = \epsilon(\beta, \mu)\epsilon(\alpha, \beta)\epsilon(\alpha, \mu) e^{\alpha+\beta+\mu}.$$

Exchanging α and β in the above computation, we immediately have

$$e_\beta e_\alpha \cdot e^\mu = \epsilon(\alpha, \mu)\epsilon(\beta, \alpha)\epsilon(\beta, \mu) e^{\beta+\alpha+\mu}$$

Multiplying both sides by $(-1)^{\langle \alpha, \beta \rangle}$ yields

$$(-1)^{\langle \alpha, \beta \rangle} e_\beta e_\alpha \cdot e^\mu = (-1)^{\langle \alpha, \beta \rangle} \epsilon(\beta, \alpha)\epsilon(\alpha, \mu)\epsilon(\beta, \mu) e^{\beta+\alpha+\mu}$$

$$= \epsilon(\alpha,\beta)\epsilon(\alpha,\mu)\epsilon(\beta,\mu)e^{\beta+\alpha+\mu}$$
$$= e_\alpha e_\beta \cdot e^\mu$$

since $\epsilon(\alpha,\beta)\epsilon(\beta,\alpha) = (-1)^{\langle\alpha,\beta\rangle}$. □

Lemma 5.3. *For the $\hat{L}/K$-module isomorphism $f : T_\chi \circ \sigma_\lambda \to T_{\chi^{(\lambda)}}$ and any $\alpha \in L$, we have $e_\alpha \circ f = (-1)^{\langle\alpha,\lambda\rangle} f \circ e_\alpha$ as operators on $\mathbb{C}[L]$.*

Proof. For any $e^\mu \in \mathbb{C}[L]$, we have

$$(-1)^{\langle\alpha,\lambda\rangle} f \circ e_\alpha \cdot e^\mu = (-1)^{\langle\alpha,\lambda\rangle} f(\epsilon(\alpha,\mu)e^{\alpha+\mu}).$$

Recall that $f(\sigma_\lambda(a)t) = af(t)$ for $a \in \hat{L}$. Thus we see that

$$e_\alpha \circ f \cdot e^\mu = f(\sigma_\lambda(e_\alpha)e^\mu) = f(\kappa^{\langle\lambda,\overline{e_\alpha}\rangle} e_\alpha e^\mu) = \kappa^{\langle\lambda,\bar{\alpha}\rangle} f(\epsilon(\alpha,\mu)e^{\alpha+\mu}) = (-1)^{\langle\alpha,\lambda\rangle} f \circ e_\alpha \cdot e^\mu.$$

□

The following Lemma is known from ([3]):

Lemma 5.4 ([3] Lemma 5.8)**.** *For any $\gamma \in L + \lambda$ and $\alpha \in L$, we have*

$$e_\alpha \circ \eta_\gamma = (-1)^{\langle\alpha,\gamma\rangle}\eta_\gamma \circ e_\alpha, \quad e_\alpha \circ \eta_\gamma = \epsilon(\alpha,\gamma)\eta_{\gamma+\alpha} = \epsilon(-\alpha,\gamma)\eta_{\gamma-\alpha}.$$

We can now define a non-trivial intertwining operator of type $\binom{V_L^{T_{\chi^{(\lambda)}}}}{V_{L+\lambda}\ V_L^{T_\chi}}$ for V_L, where $\lambda \in L^\circ$. Following [14], we define a map

$$\mathcal{Y}^{tw}_\lambda(\cdot,z) : M(1,\lambda) \to (\mathrm{End}\,(M(1)(\theta)))\{z\}, \quad v \mapsto \mathcal{Y}^{tw}_\lambda(v,z)$$

for $v = h_1(-n_1)h_2(-n_2)\cdots h_k(-n_k)\otimes e^\lambda$, where $h_i \in \mathfrak{h}$ and $n_i \geq 1$, by first defining its action on e^λ by

$$\mathcal{Y}^{tw}_\lambda(e^\lambda,z) = 2^{-\langle\lambda,\lambda\rangle} z^{-\frac{\langle\lambda,\lambda\rangle}{2}} e^{\sum_{n>0}\frac{\lambda(-n)}{n}z^n} e^{-\sum_{n>0}\frac{\lambda(n)}{n}z^{-n}},$$

where $n \in \mathbb{N} + \frac{1}{2}$. Then we define

$$W(v,z) = {}^\circ_\circ \left(\frac{1}{(n_1-1)!}\left(\frac{d}{dz}\right)^{n_1-1} h_1(z)\right)\cdots\left(\frac{1}{(n_k-1)!}\left(\frac{d}{dz}\right)^{n_k-1} h_k(z)\right)\mathcal{Y}^{tw}_\lambda(e^\lambda,z){}^\circ_\circ,$$

where, as before, the normal ordering places $h_i(n)$ with $n < 0$ to the left of $h_i(n)$ with $n > 0$. Finally, for $v \in M(1,\lambda)$ we set $\mathcal{Y}^{tw}_\lambda(v,z) = W(e^{\Delta_z}v,z)$, where

$$\Delta_z = \sum_{i=1}^{d}\sum_{m,n=0}^{\infty} c_{mn}\beta_i(m)\beta_i(n)z^{-m-n},$$

and $\{\beta_1,\beta_2,\ldots,\beta_d\}$ is an orthonormal basis of $\mathfrak{h}$, c_{mn} are the coefficients determined by the following expansion

$$-\log\left(\frac{(1+x)^{1/2}+(1+y)^{1/2}}{2}\right) = \sum_{m,n=0}^{\infty} c_{mn}x^m y^n.$$

It is known that the V_L-module $V_{L+\lambda}$ has the following decomposition

$$V_{L+\lambda} \cong \bigoplus_{\beta\in L} M(1,\beta+\lambda),$$

where $M(1,\beta+\lambda)$ are irreducible $M(1)$-modules. We also define another map by

$$\tilde{\mathcal{Y}}^{tw}_{\lambda}(u,z) = \mathcal{Y}^{tw}_{\lambda+\beta}(u,z)\otimes\eta_{\lambda+\beta}.$$

Recall that $\eta_{\lambda+\beta}$ is a linear isomorphism between T_χ and $T_{\chi^{(\lambda)}}$, while the components of $\mathcal{Y}^{tw}_{\lambda}(u,z)$ are elements of $\mathrm{End}(M(1)(\theta))\{z\}$, and $M(1)(\theta)$ can be identified with $M(1)(\theta)\otimes 1$ as a subspace of $M(1)(\theta)\otimes T_\chi = V_L^{T_\chi}$. Thus, we have the linear map

$$\tilde{\mathcal{Y}}^{tw}_{\lambda} : V_{L+\lambda} \to (\mathrm{Hom}(V_L^{T_\chi}, V_L^{T_{\chi^{(\lambda)}}}))\{z\}, u \mapsto \tilde{\mathcal{Y}}^{tw}_{\lambda}(u,z) = \mathcal{Y}^{tw}_{\lambda+\beta}(u,z)\otimes\eta_{\lambda+\beta}.$$

The next three lemmas show that $\tilde{\mathcal{Y}}^{tw}_{\lambda}$ satisfies the three conditions stated in the definition of an intertwining operator and thus is an intertwining operator of type $\binom{V_L^{T_{\chi^{(\lambda)}}}}{V_{L+\lambda}\ \ V_L^{T_\chi}}$ for V_L. From there, we show that the fusion rule $N_{V_L}\binom{V_L^{T_{\chi^{(\lambda)}}}}{V_{L+\lambda}\ \ V_L^{T_\chi}} = 1$.

Lemma 5.5. *For any $u \in V_{L+\lambda}, v \in V_L^{T_\chi}$, and any fixed $\alpha \in \mathbb{C}$, we have $u_{n+\alpha}v = 0$ for sufficiently large integer n.*

Proof. Since $v \in V_L^{T_\chi} = M(1)(\theta)\otimes T_\chi$, we have $v = w\otimes t$ for some $w \in M(1)(\theta)$ and $t \in T_\chi$. Then we have

$$\tilde{\mathcal{Y}}^{tw}_{\lambda}(u,z)v = \tilde{\mathcal{Y}}^{tw}_{\lambda}(u,z)(w\otimes t) = \mathcal{Y}^{tw}_{\lambda+\beta}(u,z)(w)\otimes\eta_{\lambda+\beta}(t).$$

However, $\mathcal{Y}^{tw}_{\lambda+\beta}$ is a nonzero intertwining operator of type $\binom{M(1)(\theta)}{M(1,\lambda+\beta)\ \ M(1)(\theta)}$ for $M(1)$ (see [3], pp.191). Then, for any $u \in M(1,\lambda+\beta) \subset V_{L+\lambda}, u_{n+\alpha}w = 0$ if n is a sufficiently large integer. □

Lemma 5.6. *Let $\alpha,\beta \in L$. For any $a \in M(1,\alpha), u \in M(1,\beta+\lambda)$, we have*

$$z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right)Y_{V_L^{T_{\chi^{(\lambda)}}}}(a,z_1)\tilde{\mathcal{Y}}^{tw}_{\lambda}(u,z_2) - z_0^{-1}\delta\left(\frac{z_2-z_1}{-z_0}\right)\tilde{\mathcal{Y}}^{tw}_{\lambda}(u,z_2)Y_{V_L^{T_\chi}}(a,z_1)$$
$$= \frac{1}{2}\sum_{p=0,1} z_2^{-1}\delta\left((-1)^p\frac{(z_1-z_0)^{1/2}}{z_2^{1/2}}\right)\tilde{\mathcal{Y}}^{tw}_{\lambda}\left(Y_{V_{L+\lambda}}(\theta^p(a),z_0)u,z_2\right),$$

where $Y_{V_L^{T_{\chi^{(\lambda)}}}}(a,z_1)$ is the vertex operator associated with $a \in M(1,\alpha) \subseteq V_L$ defined by

$$Y_{V_L^{T_{\chi^{(\lambda)}}}}(\cdot,z_1) : M(1,\alpha) \subseteq V_L \to (End(V_L^{T_{\chi^{(\lambda)}}}))\{z_1\}, \quad a \mapsto Y_{V_L^{T_{\chi^{(\lambda)}}}}(a,z_1).$$

Proof. By [14], this lemma gives the Jacobi identity for twisted vertex operators. Recall the map $\tilde{\mathcal{Y}}^{tw}_{\lambda} : M(1,\lambda+\beta) \subseteq V_{L+\lambda} \to (\mathrm{Hom}(V_L^{T_\chi}, V_L^{T_{\chi^{(\lambda)}}}))\{z\}$. Take $\lambda = 0$ and $\beta = \alpha$, then we have

$$\tilde{\mathcal{Y}}^{tw}_{0} : M(1,\alpha) \subseteq V_L \to (\mathrm{End}(V_L^{T_\chi}))\{z\} \tag{5.6.1}$$

For any $w\otimes t \in M(1)(\theta)\otimes T_\chi (= V_L^{T_\chi})$, we have

$$\begin{aligned}\tilde{\mathcal{Y}}_0^{tw}(a,z_1)(w\otimes t) &= (\mathcal{Y}_{0+\alpha}^{tw}(a,z_1)\otimes\eta_{0+\alpha})(w\otimes t)\\ &= \mathcal{Y}_\alpha^{tw}(a,z_1)(w)\otimes\eta_\alpha(t)\\ &= \mathcal{Y}_\alpha^{tw}(a,z_1)(w)\otimes e_\alpha(t)\\ &= (\mathcal{Y}_\alpha^{tw}(a,z_1)\otimes e_\alpha)(w\otimes t).\end{aligned}\tag{5.6.2}$$

The equality (5.6.2) follows from the fact that $\eta_\alpha = \eta_{0+\alpha} = \epsilon(-\alpha,0)e_\alpha\circ f = e_\alpha\circ f = e_\alpha$ since f is an isomorphism of T_χ. However, by (5.6.1), the map $\tilde{\mathcal{Y}}_0^{tw}(a,z_1)$ is the twisted vertex operator associated with $a\in M(1,\alpha)\subseteq V_L$, that is, $Y_{V_L^{T_\chi(\lambda)}}(a,z_1) = \tilde{\mathcal{Y}}_0^{tw}(a,z_1) = \mathcal{Y}_\alpha^{tw}(a,z_1)\otimes e_\alpha$. By the same argument, we have $Y_{V_L^{T_\chi}}(a,z_1) = \tilde{\mathcal{Y}}_\alpha^{tw}(a,z_1)\otimes e_\alpha$.

REMARK. Recall the map

$$\mathcal{Y}_{\alpha,\lambda+\beta}(\cdot,z_0) : M(1,\alpha)\to(\mathrm{Hom}(M(1,\lambda+\beta),M(1,\alpha+\lambda+\beta)))\{z_0\},$$

where $M(1,\alpha)\subseteq V_L, M(1,\lambda+\beta)\subseteq V_{L+\lambda}$, and $M(1,\alpha+\lambda+\beta)\subseteq V_{L+\lambda}$. This map satisfies the Jacobi identity and the $L(-1)$-derivative property. Therefore, it is the map giving a V_L-module structure for $V_{L+\lambda}$. As a result, we obtain $\mathcal{Y}_{\alpha,\lambda+\beta}(a,z_0) = Y_{V_{L+\lambda}}(a,z_0)$.

The left-hand side of the Jacobi identity is

$$\begin{aligned}&z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right)Y_{V_L^{T_\chi(\lambda)}}(a,z_1)\tilde{\mathcal{Y}}_\lambda^{tw}(u,z_2) - z_0^{-1}\delta\left(\frac{z_2-z_1}{-z_0}\right)\tilde{\mathcal{Y}}_\lambda^{tw}(u,z_2)Y_{V_L^{T_\chi}}(a,z_1)\\
&= z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right)\left(\mathcal{Y}_\alpha^{tw}(a,z_1)\otimes e_\alpha\right)\tilde{\mathcal{Y}}_\lambda^{tw}(u,z_2)\\
&\qquad - z_0^{-1}\delta\left(\frac{z_2-z_1}{-z_0}\right)\tilde{\mathcal{Y}}_\lambda^{tw}(u,z_2)\left(\mathcal{Y}_\alpha^{tw}(a,z_1)\otimes e_\alpha\right)\\
&= z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right)\left(\mathcal{Y}_\alpha^{tw}(a,z_1)\otimes e_\alpha\right)\left(\mathcal{Y}_{\lambda+\beta}^{tw}(u,z_2)\otimes\eta_{\lambda+\beta}\right)\\
&\qquad - z_0^{-1}\delta\left(\frac{z_2-z_1}{-z_0}\right)\left(\mathcal{Y}_{\lambda+\beta}^{tw}(u,z_2)\otimes\eta_{\lambda+\beta}\right)\left(\mathcal{Y}_\alpha^{tw}(a,z_1)\otimes e_\alpha\right)\\
&= z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right)\left(\mathcal{Y}_\alpha^{tw}(a,z_1)\mathcal{Y}_{\lambda+\beta}^{tw}(u,z_2)\right)\otimes\left(e_\alpha\circ\eta_{\lambda+\beta}\right)\\
&\qquad - z_0^{-1}\delta\left(\frac{z_2-z_1}{-z_0}\right)\left(\mathcal{Y}_{\lambda+\beta}^{tw}(u,z_2)\mathcal{Y}_\alpha^{tw}(a,z_1)\right)\otimes\left(\eta_{\lambda+\beta}\circ e_\alpha\right)\\
&= z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right)\left(\mathcal{Y}_\alpha^{tw}(a,z_1)\mathcal{Y}_{\lambda+\beta}^{tw}(u,z_2)\right)\otimes\left(e_\alpha\circ\eta_{\lambda+\beta}\right)\\
&\qquad - z_0^{-1}\delta\left(\frac{z_2-z_1}{-z_0}\right)\left(\mathcal{Y}_{\lambda+\beta}^{tw}(u,z_2)\mathcal{Y}_\alpha^{tw}(a,z_1)\right)\otimes\left((-1)^{(\alpha,\lambda+\beta)}e_\alpha\circ\eta_{\lambda+\beta}\right)\\
&= \left\{z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right)\mathcal{Y}_\alpha^{tw}(a,z_1)\mathcal{Y}_{\lambda+\beta}^{tw}(u,z_2)\right.\\
&\qquad \left. - (-1)^{(\alpha,\lambda+\beta)}z_0^{-1}\delta\left(\frac{z_2-z_1}{-z_0}\right)\mathcal{Y}_{\lambda+\beta}^{tw}(u,z_2)\mathcal{Y}_\alpha^{tw}(a,z_1)\right\}\otimes\left(e_\alpha\circ\eta_{\lambda+\beta}\right)\end{aligned}$$

$$= \left\{ \frac{1}{2} \sum_{p=0,1} z_2^{-1} \delta \left((-1)^p \frac{(z_1 - z_0)^{1/2}}{z_2^{1/2}} \right) \mathcal{Y}^{tw}_{\lambda+\beta+(-1)^p\alpha} \left(\mathcal{Y}_{(-1)^p\alpha,\lambda+\beta}(\theta^p(a), z_0)u, z_2 \right) \right\}$$

$$\otimes \left(e_\alpha \circ \eta_{\lambda+\beta} \right)$$

$$= \frac{1}{2} z_2^{-1} \delta \left(\frac{(z_1 - z_0)^{1/2}}{z_2^{1/2}} \right) \mathcal{Y}^{tw}_{\lambda+\beta+\alpha} \left(\mathcal{Y}_{\alpha,\lambda+\beta}(a, z_0)u, z_2 \right) \otimes \left(e_\alpha \circ \eta_{\lambda+\beta} \right)$$

$$+ \frac{1}{2} z_2^{-1} \delta \left(-\frac{(z_1 - z_0)^{1/2}}{z_2^{1/2}} \right) \mathcal{Y}^{tw}_{\lambda+\beta-\alpha} \left(\mathcal{Y}_{-\alpha,\lambda+\beta}(\theta(a), z_0)u, z_2 \right) \otimes \left(e_\alpha \circ \eta_{\lambda+\beta} \right)$$

(5.6.3)

$$= \frac{1}{2} z_2^{-1} \delta \left(\frac{(z_1 - z_0)^{1/2}}{z_2^{1/2}} \right) \mathcal{Y}^{tw}_{\lambda+\beta+\alpha} \left(Y_{V_{L+\lambda}}(a, z_0)u, z_2 \right) \otimes \left(\epsilon(\alpha, \lambda+\beta)\eta_{\lambda+\beta+\alpha} \right)$$

$$+ \frac{1}{2} z_2^{-1} \delta \left(-\frac{(z_1 - z_0)^{1/2}}{z_2^{1/2}} \right) \mathcal{Y}^{tw}_{\lambda+\beta-\alpha} \left(Y_{V_{L+\lambda}}(\theta(a), z_0)u, z_2 \right) \otimes \left(\epsilon(-\alpha, \lambda+\beta)\eta_{\lambda+\beta-\alpha} \right)$$

$$= \frac{1}{2} z_2^{-1} \delta \left(\frac{(z_1 - z_0)^{1/2}}{z_2^{1/2}} \right) \mathcal{Y}^{tw}_{\lambda+(\beta+\alpha)} \left(Y_{V_{L+\lambda}}(a, z_0)u, z_2 \right) \otimes \eta_{\lambda+(\beta+\alpha)}$$

$$+ \frac{1}{2} z_2^{-1} \delta \left(-\frac{(z_1 - z_0)^{1/2}}{z_2^{1/2}} \right) \mathcal{Y}^{tw}_{\lambda+(\beta-\alpha)} \left(Y_{V_{L+\lambda}}(\theta(a), z_0)u, z_2 \right) \otimes \eta_{\lambda+(\beta-\alpha)}$$

$$= \frac{1}{2} z_2^{-1} \delta \left(\frac{(z_1 - z_0)^{1/2}}{z_2^{1/2}} \right) \tilde{\mathcal{Y}}^{tw}_{\lambda} \left(Y_{V_{L+\lambda}}(a, z_0)u, z_2 \right)$$

$$+ \frac{1}{2} z_2^{-1} \delta \left(-\frac{(z_1 - z_0)^{1/2}}{z_2^{1/2}} \right) \tilde{\mathcal{Y}}^{tw}_{\lambda} \left(Y_{V_{L+\lambda}}(\theta(a), z_0)u, z_2 \right)$$

$$= \frac{1}{2} \sum_{p=0,1} z_2^{-1} \delta \left((-1)^p \frac{(z_1 - z_0)^{1/2}}{z_2^{1/2}} \right) \tilde{\mathcal{Y}}^{tw}_{\lambda} \left(Y_{V_{L+\lambda}}(\theta^p(a), z_0)u, z_2 \right).$$

Lines (5.6.3) follows from the Remark on the previous page. This completes the proof of the Jacobi identity. □

Lemma 5.7. *The map $\tilde{\mathcal{Y}}^{tw}_{\lambda}$ satisfies the $L(-1)$-derivative property*

$$\tilde{\mathcal{Y}}^{tw}_{\lambda}(L(-1)u, z) = \frac{d}{dz} \tilde{\mathcal{Y}}^{tw}_{\lambda}(u, z).$$

Proof. Let $u \in M(1, \lambda+\beta) \subseteq V_{L+\lambda}$, then it follows that

$$\begin{aligned}
\tilde{\mathcal{Y}}^{tw}_{\lambda}(L(-1)u, z) &= \mathcal{Y}^{tw}_{\lambda+\beta}(L(-1)u, z) \otimes \eta_{\lambda+\beta} \\
&= \left(\frac{d}{dz} \mathcal{Y}^{tw}_{\lambda+\beta}(u, z) \right) \otimes \eta_{\lambda+\beta} \\
&= \frac{d}{dz} \left(\mathcal{Y}^{tw}_{\lambda+\beta}(u, z) \otimes \eta_{\lambda+\beta} \right) \\
&= \frac{d}{dz} \tilde{\mathcal{Y}}^{tw}_{\lambda}(u, z),
\end{aligned}$$

where the second equality follows from Proposition 9.4.3 of [14]. □

Since $\tilde{\mathcal{Y}}_\lambda^{tw}$ is a non-trivial intertwining operator of type $\binom{V_L^{T_{\chi^{(\lambda)}}}}{V_{L+\lambda}\ V_L^{T_\chi}}$ for V_L, we have

$$N\binom{V_L^{T_{\chi^{(\lambda)}}}}{V_{L+\lambda}\ V_L^{T_\chi}} \geq 1.$$

However, Proposition 3.2.2 (1) and Proposition 2.5 together imply that

$$N\binom{V_L^{T_{\chi^{(\lambda)}}}}{V_{L+\lambda}\ V_L^{T_\chi}} = 1. \tag{5.7.1}$$

Thus, by Lemma 5.1, we have shown

Theorem 5.8. *For any $\lambda \in S$ and any irreducible $\hat{L}/K$-module T_χ, we have $V_{L+\lambda} \boxtimes V_L^{T_\chi} = V_L^{T_{\chi^{(\lambda)}}}$, where $T_{\chi^{(\lambda)}}$ is an irreducible $\hat{L}/K$-module such that $\chi^{(\lambda)}(a) = (-1)^{\langle\lambda,\bar{a}\rangle}\chi(a)$ for any $a \in \hat{L}/K$.*

6. Fusion products $V_L^{T_{\chi_1}} \boxtimes V_L^{T_{\chi_2}}$

In this section we compute the fusion product of two V_L-modules of twisted type. Let M^i run over the set of equivalence classes of irreducible V_L-modules, then by the definition of fusion product, we have

$$V_L^{T_{\chi_1}} \boxtimes V_L^{T_{\chi_2}} = \sum_{\lambda\in S} N_{V_L}\binom{V_{L+\lambda}}{V_L^{T_{\chi_1}} V_L^{T_{\chi_2}}} V_{L+\lambda} + \sum_{V_L^{T_{\chi_j}}} N_{V_L}\binom{V_L^{T_{\chi_j}}}{V_L^{T_{\chi_1}} V_L^{T_{\chi_2}}} V_L^{T_{\chi_j}},$$

where $S = \{\lambda_1, \ldots, \lambda_k\}$ is a set of representatives of equivalence classes of L in its dual lattice L° and $V_L^{T_{\chi_j}}$ runs over the equivalence classes of irreducible θ-twisted V_L-modules. We begin by quoting here only a part of an important theorem from [3].

Theorem 6.1 ([3], Theorem 5.1)**.** *Let L be a positive-definite even lattice. For any irreducible V_L^+-modules M^i $(i = 1, 2, 3)$, the fusion rule of type $\binom{M^3}{M^1 M^2}$ is either 0 or 1. The fusion rule of type $\binom{M^3}{M^1 M^2}$ is 1 if and only if the M^i satisfy one of the following conditions:*

(a) *$M^1 = V_L^{T_\chi,+}$ for an irreducible $\hat{L}/K$-module T_χ and (M^2, M^3) is one of the following pairs: $(V_{L+\lambda}, V_L^{T_{\chi^{(\lambda)}},\pm}), ((V_L^{T_{\chi^{(\lambda)}},\pm})', (V_{L+\lambda})')$ for $\lambda \in L^o$ such that $2\lambda \notin L$.*

(b) *$M^1 = V_L^{T_\chi,-}$ for an irreducible $\hat{L}/K$-module T_χ and (M^2, M^3) is one of the following pairs: $(V_{L+\lambda}, V_L^{T_{\chi^{(\lambda)}},\pm}), ((V_L^{T_{\chi^{(\lambda)}},\pm})', (V_{L+\lambda})')$ for $\lambda \in L^o$ such that $2\lambda \notin L$.*

We now show the first lemma of this section.

Lemma 6.2. *Let $\lambda \in S$. If χ_1 and χ_2 are central characters of $\hat{L}/K$ such that $\chi_2(a) = (-1)^{\langle\bar{a},\lambda\rangle}\chi_1(a)$ for any $a \in \hat{L}$, then we have*

$$N_{V_L}\binom{V_{L+\lambda}}{V_L^{T_{\chi_1}}\ V_L^{T_{\chi_2}}} = 1.$$

Proof. By Theorem 6.1 (a), for any $\lambda \in L$ such that $2\lambda \notin L$ and $\chi_2(a) = (-1)^{\langle \bar{a}, \lambda \rangle}\chi_1(a)$, for any $a \in \hat{L}$, we have

$$N_{V_L^+}\begin{pmatrix} (V_{L+\lambda})' \\ V_L^{T_{\chi_1},+} \quad (V_L^{T_{\chi_2},+})' \end{pmatrix} = 1.$$

By Proposition 3.7 of [3], one can verify that $(V_{L+\lambda})' \cong V_{L-\lambda}$ and $(V_L^{T_{\chi_2},+})' \cong V_L^{T_{\chi_2'},+}$, where $\chi_2'(a) = (-1)^{\langle \bar{a}, \bar{a} \rangle/2}\chi_2(a)$ for any $a \in \hat{L}$. Therefore we have

$$N_{V_L^+}\begin{pmatrix} V_{L-\lambda} \\ V_L^{T_{\chi_1},+} \quad V_L^{T_{\chi_2'},+} \end{pmatrix} = 1.$$

Proposition 2.5 now shows

$$N_{V_L}\begin{pmatrix} V_{L-\lambda} \\ V_L^{T_{\chi_1}} \quad V_L^{T_{\chi_2'}} \end{pmatrix} \leq N_{V_L^+}\begin{pmatrix} V_{L-\lambda} \\ V_L^{T_{\chi_1},+} \quad V_L^{T_{\chi_2'},+} \end{pmatrix} = 1.$$

By the well-known symmetries of fusion rules (Proposition 2.4), it follows that

$$\begin{aligned} N_{V_L}\begin{pmatrix} V_{L-\lambda} \\ V_L^{T_{\chi_1}} \quad V_L^{T_{\chi_2'}} \end{pmatrix} &= N_{V_L}\begin{pmatrix} (V_L^{T_{\chi_2'}})' \\ V_L^{T_{\chi_1}} \quad (V_{L-\lambda})' \end{pmatrix} \\ &= N_{V_L}\begin{pmatrix} V_L^{T_{\chi_2''}} \\ V_L^{T_{\chi_1}} \quad V_{L+\lambda} \end{pmatrix} \\ &= N_{V_L}\begin{pmatrix} V_L^{T_{\chi_2}} \\ V_L^{T_{\chi_1}} \quad V_{L+\lambda} \end{pmatrix} \qquad (6.2.1) \\ &= N_{V_L}\begin{pmatrix} V_L^{T_{\chi_2}} \\ V_{L+\lambda} \quad V_L^{T_{\chi_1}} \end{pmatrix} \\ &= 1. \end{aligned}$$

In the computation above, the equality (6.2.1) follows from $\chi_2''(a) = (-1)^{(\bar{a}, \bar{a})/2}\chi_2'(a) = (-1)^{(\bar{a}, \bar{a})/2}(-1)^{(\bar{a}, \bar{a})/2}\chi_2(a) = \chi_2(a)$. □

Lemma 6.3. *Let χ_1 and χ_2 be central characters of $\hat{L}/K$ such that $\chi_2(a) = (-1)^{(\bar{a}, \lambda)}\chi_1(a)$ for any $a \in \hat{L}$ and χ_i any central character of $\hat{L}/K$ such that $\chi_i(\kappa) = -1$. Then we have*

$$N_{V_L}\begin{pmatrix} V_L^{T_{\chi_i}} \\ V_L^{T_{\chi_1}} \quad V_L^{T_{\chi_2}} \end{pmatrix} = 0.$$

Proof. Let $\varepsilon_i \in \{\pm\}$ and $i \in \{1, 2\}$, then

$$\begin{aligned} N_{V_L}\begin{pmatrix} V_L^{T_{\chi_i}} \\ V_L^{T_{\chi_1}} \quad V_L^{T_{\chi_2}} \end{pmatrix} &\leq N_{V_L^+}\begin{pmatrix} V_L^{T_{\chi_i}} \\ V_L^{T_{\chi_1},\varepsilon_1} \quad V_L^{T_{\chi_2},\varepsilon_2} \end{pmatrix} \\ &= N_{V_L^+}\begin{pmatrix} (V_L^{T_{\chi_2},\varepsilon_2})' \\ V_L^{T_{\chi_1},\varepsilon_1} \quad (V_L^{T_{\chi_i}})' \end{pmatrix} \end{aligned}$$

$$= N_{V_L^+}\begin{pmatrix} V_L^{T_{\chi_2'},\varepsilon_2} \\ V_L^{T_{\chi_1},\varepsilon_1} \quad V_L^{T_{\chi_i'}} \end{pmatrix}$$
$$\leq N_{V_L^+}\begin{pmatrix} V_L^{T_{\chi_2'},\varepsilon_2} \\ V_L^{T_{\chi_1},\varepsilon_1} \quad V_L^{T_{\chi_i'},\varepsilon_i} \end{pmatrix} = 0$$

since all three are of twisted type (see Proposition 3.2.2 (2)). □

Hence, we have shown

Theorem 6.4. *If χ_1 and χ_2 are central characters of $\hat{L}/K$, then $V_L^{T_{\chi_1}} \boxtimes V_L^{T_{\chi_2}} = \sum_\lambda V_{L+\lambda}$, where $\lambda \in L^\circ/L$ such that $\chi_2(a) = (-1)^{\langle \bar{a},\lambda\rangle}\chi_1(a)$ for any $a \in \hat{L}$.*

ACKNOWLEDGEMENTS. The author wishes to thank Prof. Chongying Dong and Prof. Kiyokazu Nagatomo for their expert guidance and unwavering support. The author would also like to thank the referee for his many insightful comments.

References

[1] T. Abe: *Fusion Rules for the Free Bosonic Orbifold Vertex Operator Algebra*, J. Algebra, **229** (2000), 333–374.
[2] T. Abe: *Fusion Rules for the Charge Conjugation Orbifold*, J. Algebra, **242** (2001), 624–655.
[3] T. Abe, C. Dong and H-S. Li: *Fusion Rules for the Vertex Operator Algebras $M(1)^+$ and V_L^+*, Comm. Math. Phys. **253** (2005), 171–219.
[4] R. Borcherds: *Vertex Algebras, Kac-Moody Algebras, and the Monster*, Proc. Nal. Acad. Sci., USA **83** (1986), 3068–3071.
[5] R. Blumenhagen and E. Plauschinn: Introduction to Conformal Field Theory, Lecture Notes in Physics **779**, Springer, Dordrecht, 2009.
[6] C. Dong: *Vertex Algebras Associated with Even Lattices*, J. Algebra, **160** (1993), 245–265.
[7] C. Dong: *Twisted Modules for Vertex Algebras Associated with Even Lattices*, J. Algebra, **165** (1994), 91–112.
[8] C. Dong, X. Jiao and F. Xu: *Quantum Dimensions and Quantum Galois Theory*, Trans. Amer. Math. Soc. **365** (2013), 6441–6469.
[9] C. Dong and J. Lepowsky: Generalized Vertex Algebras and Relative Vertex Operators, Progress in Math. **112**, Birkhäuser, Boston, 1993.
[10] C. Dong, H. Li and G. Mason: *Twisted Representations of Vertex Operator Algebras*, Math. Ann. **310** (1998), 571–600.
[11] C. Dong and K. Nagatomo: *Classification of Irreducible Modules for the Vertex Operator Algebras $M(1)^+$*, J. Algebra, **216** (1999), 384–404.
[12] C. Dong and K. Nagatomo: *Representations of Vertex Operator Algebras V_L^+ for Rank One Lattice L*, Comm. Math. Phys. **202** (1999), 169–195.
[13] I. Frenkel, Y. Huang and J. Lepowsky: On Axiomatic Approaches to Vertex Operator Algebras and Modules, Mem. Amer. Math. Soc. **104** (1993), no. 494.
[14] I. Frenkel, J. Lepowsky and A. Meurman: Vertex Operator Algebras and the Monster, Pure and Appl. Math **134**, Academic Press, Boston, 1988.
[15] T. Gannon: Moonshine beyond the Monster: The Bridge Connecting Algebra, Modular Forms and Physics, Cambridge Monographs on Mathematical Phycics, Cambridge University Press, Cambridge, 2006.
[16] Y. Huang: *Intertwining Operators among Twisted Modules Associated to Not-Necessarily-Commuting Automorphisms*, J. Algebra, **493** (2018), 346–380.

[17] Y. Huang and J. Lepowsky: *A Theory of Tensor Products for Module Categories for a Vertex Operator Algebra, I*, Selecta Math. (N.S.) **1** (1995) 699–756.
[18] K. Iohara and Y. Koga: Representation Theory of the Virasoro Algebra, Springer Monographs in Mathematics, Springer-Verlag, London Ltd, London, 2011.
[19] H. Li: *Representation Theory and Tensor Product Theory for Modules for a Vertex Operator Algebra*, Ph.D Thesis, Rutgers University, 1994.
[20] J. Lepowsky and H. Li: Introduction to Vertex Operator Algebras and Their Representations, Progress in Math. **227**, Birkhuauser, Boston, 2003.

Sweet Briar College
Sweet Briar, VA 24595
USA
e-mail: dnguyen@sbc.edu