Minimization problems on the Hardy-Sobolev inequality

Masato Hashizume

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Minimization problems on the Hardy-Sobolev inequality

Masato Hashizume

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Abstract We study minimization problems on Hardy-Sobolev type inequality. We consider the case where singularity is in interior of bounded domain $\Omega \subseteq \mathbb{R}^N$. The attainability of best constants for Hardy-Sobolev type inequalities with boundary singularities have been studied so far, for example [5] [6] [9] etc. . . . According to their results, the mean curvature of $\partial \Omega$ at singularity affects the attainability of the best constants. In contrast with the case of boundary singularity, it is well known that the best Hardy-Sobolev constant

$$
\mu_s(\Omega) := \left\{ \int_\Omega |\nabla u|^2 dx \left| u \in H^1_0(\Omega), \int_\Omega \frac{|u|^{2^*(s)}}{|x|^s} = 1 \right. \right\}
$$

is never achieved for all bounded domain $\Omega$ if $0 \in \Omega$. We see that the position of singularity on domain is related to the existence of minimizer. In this paper, we consider the attainability of the best constant for the embedding $H^1(\Omega) \hookrightarrow L^{2^*(s)}(\Omega)$ for bounded domain $\Omega$ with $0 \in \Omega$. In this problem, scaling invariance doesn’t hold and we can not obtain information of singularity like mean curvature.

Keywords critical exponent · Hardy-Sobolev inequality · minimization problem · Neumann

Mathematics Subject Classification (2000) 35J20

1 Introduction

We study the minimization problems for the Hardy-Sobolev type inequalities. Let $N \geq 3$, $\Omega$ is bounded domain in $\mathbb{R}^N$, $0 \in \Omega$, $0 < s < 2$, and $2^*(s) := 2(N-s)/(N-2)$. The Hardy-Sobolev inequality asserts that there exists a positive constant $C$ such that

$$
C \left( \int_\Omega \frac{|u|^{2^*(s)}}{|x|^s} dx \right)^{\frac{2}{2^*(s)}} \leq \int_\Omega |\nabla u|^2 dx
$$

(1)
for all \( u \in H^1_0(\Omega) \). For \( s = 0 \), the inequality (1) is called Sobolev inequality and for \( s = 2 \), the inequality (1) is called Hardy inequality.

In the non-singular case \( (s = 0) \), it is well known that the best Sobolev constant \( S \) is independent of domain \( \Omega \) and \( S \) is never achieved for all bounded domains. But if \( \Omega = \mathbb{R}^N \) and \( H^1(\Omega) \) is replaced by the function space of \( u \in L^{2N/(N-2)}(\Omega) \) with \( \mathcal{V} u \in L^2(\Omega) \), then \( S \) is achieved by the function \( u(x) = c(1 + |x|^2)^{(2-N)/2} \) and hence the value \( S = N(N-2)\pi^{\frac{2}{N}}(\Gamma(N/2)/\Gamma(N))^{2/N} \) explicitly (see [1], [13] and [16]).

In the case of \( s = 2 \), the best constant for the Hardy inequality is \( [(N-2)/2]^2 \) and this constant is never achieved for all bounded domains and \( \mathbb{R}^N \). This fact suggests that it is possible to improve this inequality. For example Brezis and Vazquez [2], many people research the optimal inequality of (1). In other words, the best remainder term for (1) is studied actively.

In the case of \( 0 < s < 2 \), the best Hardy-Sobolev constant is defined by

\[
\mu_s(\Omega) := \left\{ \int_{\Omega} |\nabla u|^2 |u|^{2s} \right\}^{\frac{1}{2}} u \in H^1_0(\Omega), \quad \int_{\Omega} |u|^{2s} \right\}
\]

This constant has some similar properties to these of the best Sobolev constant. Indeed, due to scaling invariance, \( \mu_s(\Omega) \) is independent of \( \Omega \), and thus \( \mu_s := \mu_s(\Omega) = \mu_s(\mathbb{R}^N) \) is not attained for all bounded domains. If \( \Omega = \mathbb{R}^N \), then \( \mu_s \) is attained by

\[
y_a(x) = \left[ a(N-s)(N-2) \right]^{\frac{N-2}{2s-N}} (a + |x|^{2-s})^{\frac{2-N}{2s-N}}
\]

for some \( a > 0 \) and hence

\[
\mu_s = (N-2)(N-s) \left( \frac{\omega_{N-1}}{2-s} \right)^{\frac{2}{N-s}} \Gamma\left( \frac{N-s}{2-s} \right)
\]

(see [9] and [13]) where \( \omega_{N-1} \) is the area of the unit sphere in \( \mathbb{R}^N \).

On the other hand, for \( 0 \in \partial \Omega \), the result of the attainability for \( \mu_s(\Omega) \) is quite different from that in the situation of \( 0 \in \Omega \). By Ghoussoub-Robert [6], it has proved that if \( \Omega \) has smooth boundary and the mean curvature of \( \partial \Omega \) at 0 is negative, then the extremal of \( \mu_s(\Omega) \) exists for all \( N \geq 3 \). Recently, Lin and Wadade [14] have studied the following minimization problem:

\[
\mu_{s,p}^\lambda(\Omega) := \inf \left\{ \int_{\Omega} |\nabla u|^2 dx + \lambda \left( \int_{\Omega} |u|^p dx \right)^\frac{2}{p} \right\} \quad u \in H^1_0(\Omega), \quad \int_{\Omega} \frac{|u|^{2s} dx}{|x|^s} = 1
\]

where \( \lambda \in \mathbb{R} \) and \( 2 \leq p \leq 2N/(N-2) \). Furthermore, as related results, Hsia, Lin and Wadade [10] studied the existence of the solution of double critical elliptic equations related with \( \mu_{s,2s}^\lambda(\Omega) \), that is, they have showed the existence of the solution for

\[
\begin{aligned}
-\Delta u + \lambda u^{2^*-1} + \frac{|u|^{2^*(s)-1}}{|x|^s} &= 0, & & u > 0, & & \text{in } \Omega \\
u &= 0 & & & & \text{on } \partial \Omega
\end{aligned}
\]

under the appropriate conditions where \( 2^* = 2N/(N-2) \). To prove these results, we use the theorem of Egnell [4]. He showed that the existence of the extremal for \( \mu_s(\Omega) \) if \( \Omega \) is a half space \( \mathbb{R}^N_+ \) or an open cone. The open cone \( \mathcal{C} \) is written of the form \( \mathcal{C} := \{ x \in \mathbb{R}^N | x = \ldots \} \).
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where $\Sigma$ is connected domain on the unit sphere $\mathbb{S}^{N-1}$ in $\mathbb{R}^N$. By this result, $\mu_s(C) > \mu_s(\mathbb{R}^N)$ and there is a positive solution for

$$\begin{cases}
-\Delta u = \frac{|u|^{2^*(s)-2}}{|x|^{s}} & \text{in } \mathcal{C}, \\
u = 0 & \text{on } \partial \mathcal{C}, \quad \text{and } u(x) = o(|x|^{2-N}) \text{ as } x \to \infty.
\end{cases}$$

The Neumann case also has been studied. Let $\Omega$ has $C^2$ boundary and the mean curvature of $\partial \Omega$ at 0 is positive. Ghoussoub and Kang [5] have showed that there is a least energy solution for

$$\begin{cases}
-\Delta u + \lambda u = \frac{|u|^{2^*(s)-1}}{|x|^{s}} & \text{in } \Omega, \\
\frac{\partial u}{\partial \nu} = 0 & \text{on } \partial \Omega
\end{cases}$$

for $N \geq 3, \lambda > 0$.

Like these results, if $0 \in \partial \Omega$, we can use the benefit of the mean curvature of $\partial \Omega$ at 0 to show the results. However if $0 \notin \Omega$, we cannot obtain the information of singularity such the mean curvature, and the fact causes some technical difficulties.

In this paper, we consider the attainability for the following minimization problem

$$\mu_s^N(\Omega) := \inf \left\{ \int_{\Omega} (|\nabla u|^2 + u^2) \, dx \left| \begin{array}{l} u \in H^1(\Omega), \\
\int_{\Omega} \frac{|u|^{2^*(s)}}{|x|^s} \, dx = 1 \end{array} \right. \right\}.$$

The main theorem is as follows.

**Theorem 1** Let $\partial \Omega$ has a smoothness which the Sobolev embeddings hold, then the following statements hold true.

(I) If $\Omega$ is sufficiently small, then $\mu_s^N(\Omega)$ is attained. Especially, if $\Omega$ satisfies the following:

$$|\Omega| \left( \int_{\Omega} |x|^{-s} \, dx \right)^{-\frac{2}{2^*(s)}} \leq \mu_s$$

then $\mu_s^N(\Omega)$ is attained, where $|\Omega|$ is the $N$-dimensional Lebesgue measure of domain $\Omega$.

(II) There is a positive constant $M$ which depends on only $\Omega$ such that $\mu_s^N(r\Omega)$ is never attained if $r > M$.

Eventually, the size of domain affects the attainability of $\mu_s^N(\Omega)$.

The rest of the paper is organized as follows. In Section 2 we introduce three lemmas to prove Theorem 1. Then in Section 3 we prove Theorem 1 using the lemmas in Section 2. In Section 4, as an application, we consider the case when singularity is in the boundary of domain. Then we introduce a new result concerning the attainability of $\mu_s^N(\Omega)$ with boundary singularity.

**2 Preparation**

In this section, we prepare some lemmas to prove Theorem 1.
Lemma 1 For \( r > 0 \), the value \( \mu_{s,r}^N(\Omega) \) is defined by

\[
\mu_{s,r}^N(\Omega) := \inf \left\{ \int_{\Omega} (|\nabla u|^2 + ru^2) \, dx \middle| u \in H^1(\Omega), \int_{\Omega} \frac{|u|^{2^*(s)}}{|x|^s} \, dx = 1 \right\}.
\]

We have

\[
\mu_{s,r}^N(\Omega) = \mu_{s}^N(\Omega).
\]

Proof For \( r > 0 \) and \( u \in H^1(\Omega) \), \( u_r \) is defined by the scaling of \( u \), that is \( u_r(x) := r^{\frac{2^* - N}{2s}} u(x/r) \in H^1(r\Omega) \). Note that

\[
\int_{r\Omega} |\nabla u_r|^2 \, dx = \int_{\Omega} |\nabla u|^2 \, dx,
\]

\[
\int_{r\Omega} u_r^2 \, dx = r^2 \int_{\Omega} u^2 \, dx,
\]

\[
\int_{r\Omega} \frac{u_r^{2^*(s)}}{|x|^s} \, dx = \int_{\Omega} \frac{u^{2^*(s)}}{|x|^s} \, dx.
\]

With these facts in mind, taking \( u \in H^1(\Omega) \) such that

\[
\int_{\Omega} \frac{u^{2^*(s)}}{|x|^s} \, dx = 1, \quad \int_{\Omega} (|\nabla u|^2 + r^2 u^2) \, dx \leq \mu_{s,r}^N(\Omega) + \varepsilon
\]

for \( \varepsilon > 0 \) sufficiently small, we have

\[
\mu_{s}^N(\Omega) \leq \int_{r\Omega} (|\nabla u_r|^2 + u_r^2) \, dx = \int_{\Omega} (|\nabla u|^2 + r^2 u^2) \, dx \leq \mu_{s,r}^N(\Omega) + \varepsilon.
\]

Hence we have \( \mu_{s,r}^N(\Omega) \leq \mu_{s}^N(\Omega) \).

The inverse also holds by replacing \( \Omega \) with \( r\Omega \).

Lemma 2 There exists a positive constant \( C \) which depends on only \( \Omega \) such that

\[
\mu_s \left( \int_{\Omega} \frac{|u|^{2^*(s)}}{|x|^s} \, dx \right)^{\frac{2}{2^*(s)} - \frac{1}{2}} \leq \int_{\Omega} |\nabla u|^2 \, dx + C \int_{\Omega} u^2 \, dx \quad (u \in H^1(\Omega)).
\]

Before beginning the proof, we make a remark. H. Jaber [12] has shown that the following theorem.

Theorem 2 ([12]) If \((M,g)\) is a compact Riemannian manifold without boundary and \(0 \in M\), there is a constant \( C = C(M, g) \) such that

\[
\mu_s \left( \int_{M} \frac{|u|^{2^*(s)}}{d_g(x,0)^s} \, dv_g \right)^{\frac{2}{2^*(s)}} \leq \int_{M} |\nabla u|^2 \, dv_g + C \int_{M} u^2 \, dv_g \quad (u \in H^1(M))
\]

where \( d_g \) is the Riemannian distance on \( M \).

Different from Theorem 2, \( \Omega \) is bounded domain of \( \mathbb{R}^N \) and therefore \( \Omega \) has a boundary, thus we can show the inequality (2) simply.
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Proof Let $0 \in \Omega_1 \subset \Omega_2 \subset \Omega$ and these two subdomain are taken suitable again later. A cut-off function is defined by $\phi$ which satisfies

$$\phi \in C_c^\infty(\mathbb{R}^N), \quad 0 \leq \phi \leq 1 \text{ in } \Omega, \quad \phi = 1 \text{ on } \Omega_1, \quad \phi = 0 \text{ on } \Omega \setminus \Omega_2.$$ 

Here, we construct a partition of unity $\eta_1, \eta_2$ defined by

$$\eta_1 := \frac{\phi^2}{\phi^2 + (1 - \phi)^2}, \quad \eta_2 := \frac{(1 - \phi)^2}{\phi^2 + (1 - \phi)^2}.$$

Note that $\eta_1^{\frac{1}{2}}, \eta_2^{\frac{1}{2}} \in C^2(\Omega)$ by the definition. We may assume that $u \in C^\infty(\Omega) \cap H^1(\Omega)$ by density. We have

$$\mu_s \left( \int_\Omega \frac{|u|^{2^*(s)}}{|x|^s} \, dx \right)^{\frac{2}{2^*(s)}} = \mu_s \|u^2\|_{L^{2^*(s)/2}(\Omega, |x|^{-s})} = \mu_s \left( \sum_{i=1}^2 \eta_i u^2 \right)_{L^{2^*(s)/2}(\Omega, |x|^{-s})} \leq \mu_s \sum_{i=1}^2 \|\eta_i u^2\|_{L^{2^*(s)/2}(\Omega, |x|^{-s})} \leq \mu_s \sum_{i=1}^2 \left( \int_\Omega \frac{|\eta_i^{\frac{1}{2}} u^{2^*(s)}}{|x|^s} \, dx \right)^{\frac{2}{2^*(s)}}$$

$$= \mu_s \sum_{i=1}^2 \left( \int_\Omega \frac{|\eta_i^{\frac{1}{2}} u|^{2^*(s)}}{|x|^s} \, dx \right) \leq \int_\Omega |\nabla (\eta_1^{\frac{1}{2}} u)|^2 \, dx,$$

We estimate $I_1, I_2$ for each.

For $I_1$, since $\text{supp} \eta_1 \subset \Omega$ we can use the Hardy-Sobolev inequality. We get that

$$I_1 = \mu_2 \left( \int_\Omega \frac{|\eta_1^{\frac{1}{2}} u^{2^*(s)}}{|x|^s} \, dx \right)^{\frac{2}{2^*(s)}} \leq \int_\Omega |\nabla (\eta_1^{\frac{1}{2}} u)|^2 \, dx$$

$$= \int_\Omega |\nabla u|^2 \eta_1 \, dx + \int_\Omega \nabla (\eta_1^{\frac{1}{2}} u) \cdot \nabla (\eta_1^{\frac{1}{2}} u^2) \, dx.$$ 

Since $\eta_1^{\frac{1}{2}} \in C^2(\Omega)$ we may integrate by parts the second term and hence we obtain

$$I_1 \leq \int_\Omega |\nabla u|^2 \eta_1 \, dx - \int_\Omega \Delta (\eta_1^{\frac{1}{2}}) \eta_1^{\frac{1}{2}} u^2 \, dx \quad (3)$$

For $I_2$, since $0 \not\in \text{supp} \eta_2$ and taking account to that $\eta = 0$ on $\Omega_1$ we have

$$I_2 = \mu_2 \left( \int_\Omega \frac{|\eta_2^{\frac{1}{2}} u^{2^*(s)}}{|x|^s} \, dx \right)^{\frac{2}{2^*(s)}} = \mu_2 \left( \int_{\Omega \setminus \Omega_1} \frac{|\eta_2^{\frac{1}{2}} u^{2^*(s)}}{|x|^s} \, dx \right)^{\frac{2}{2^*(s)}} \leq \mu_2 \cdot a \left( \int_{\Omega \setminus \Omega_1} \frac{|\eta_2^{\frac{1}{2}} u^{2^*(s)}}{|x|^s} \, dx \right)^{\frac{2}{2^*(s)}} \leq \mu_2 \cdot a \cdot |\Omega \setminus \Omega_1|^{\frac{2}{2^*(s)} - \frac{2}{s^*}} \int_{\Omega \setminus \Omega_1} |\eta_2^{\frac{1}{2}} u^{2^*(s)}} |^s \, dx$$

$$\leq \mu_2 \cdot a \cdot |\Omega \setminus \Omega_1|^{\frac{2}{2^*(s)} - \frac{2}{s^*}} S(\Omega, \Omega_1)^{-1} \int_{\Omega \setminus \Omega_1} |\nabla (\eta_2^{\frac{1}{2}} u)|^2 \, dx$$

$$= \mu_2 \cdot a \cdot |\Omega \setminus \Omega_1|^{\frac{2}{2^*(s)} - \frac{2}{s^*}} S(\Omega, \Omega_1)^{-1} \int_\Omega |\nabla (\eta_2^{\frac{1}{2}} u)|^2 \, dx.$$
where \( a := \text{dist}(0, \partial \Omega_1)^{-2s/2s'}(s) \) and
\[
S(\Omega, \Omega_1) := \inf \left\{ \int_{\Omega \setminus \Omega_1} |\nabla u|^2 dx \mid u \in H^1(\Omega), \ u = 0 \text{ on } \partial \Omega_1, \ \int_{\Omega \setminus \Omega_1} |u|^{2s'} = 1 \right\}.
\]

Here, let us take \( \Omega_0 \subset \Omega_1 \). It is clearly that \( a \leq \text{dist}(0, \partial \Omega_0)^{-2s/2s'}(s) \). On the other hand, for \( u \in H^1(\Omega \setminus \Omega_1) \) such that \( u = 0 \) on \( \partial \Omega_1 \), we define \( v \in H^1(\Omega \setminus \Omega_0) \) by
\[
v := \begin{cases} u & \text{in } \Omega \setminus \Omega_1 \vspace{1mm} \\ 0 & \text{in } \Omega_1 \setminus \Omega_0. \end{cases}
\]

By identifying \( u \in H^1(\Omega \setminus \Omega_1) \) with \( v \in H^1(\Omega \setminus \Omega_0) \) concerning the calculation of the Sobolev quotient, we may see that
\[
\{ u \in H^1(\Omega \setminus \Omega_1) \mid u = 0 \text{ on } \partial \Omega_1 \} \subset \{ u \in H^1(\Omega \setminus \Omega_0) \mid u = 0 \text{ on } \partial \Omega_0 \}.
\]

Hence we obtain \( S(\Omega, \Omega_1) \geq S(\Omega, \Omega_0) \). Consequently, if \( \Omega_1 \) is sufficiently large, \( a \) and \( S(\Omega, \Omega_1)^{-1} \) is bounded from above uniformly. By choosing \( \Omega_1 \) and \( \Omega_2 \) close to \( \Omega \) we obtain
\[
I_2 \leq \frac{1}{2} \int_\Omega |\nabla (\eta_2^{\frac{1}{2}} u)|^2 dx.
\]

Therefore
\[
I_2 \leq \int_\Omega |\nabla u|^2 \eta_2 dx + \int_\Omega |\nabla \eta_2^{\frac{1}{2}}|^2 u^2 dx. \quad (4)
\]

Here, since \( \eta_1^{\frac{1}{2}}, \ \eta_2^{\frac{1}{2}} \in C^2(\Omega) \) there is a positive constant \( C \) such that
\[
\max_{x \in \Omega} |\Delta \eta_1^{\frac{1}{2}}| \leq \frac{C}{2}, \quad \max_{x \in \Omega} |\nabla \eta_2^{\frac{1}{2}}|^2 \leq \frac{C}{2}. \quad (5)
\]

This constant depends on only \( \Omega \).

Consequently (3), (4) and (5) yield that
\[
\mu_s \left( \int_\Omega \frac{|u|^{2s}(s)}{|x|^s} dx \right)^{\frac{2}{2s'}} \leq I_1 + I_2 \leq \int_\Omega |\nabla u|^2 dx + C \int_\Omega u^2 dx.
\]

**Lemma 3** \( \mu_s^N(\Omega) \leq \mu_s \) holds (see [9], Lemma 11.1). Furthermore, the following statements hold true:

(I) If \( \mu_s^N(\Omega) < \mu_s \), then \( \mu_s^N(\Omega) \) is attained.

(II) If \( \mu_s^N(\Omega) = \mu_s \), then \( \mu_s^N(r\Omega) \) is not attained for all \( r > 1 \).

Firstly, we prove Lemma 3 (I).

**Proof (Proof of Lemma 3 (I))**

Assume \( \{ u_n \}_{n=1}^\infty \subset H^1(\Omega) \) is a minimizing sequence of \( \mu_s^N(\Omega) \). Without loss of generality, we may assume
\[
\int_\Omega \frac{|u_n|^{2s}(s)}{|x|^s} dx = 1 \quad (6)
\]

for all \( n \in \mathbb{N} \) and which implies
\[
\int_\Omega (|\nabla u_n|^2 + u_n^2) dx = \mu_s^N(\Omega) + o(1) \quad (n \to \infty). \quad (7)
\]
Thus $u_n$ is bounded in $H^1(\Omega)$. So we can suppose, up to a subsequence,

$$
u_n \rightharpoonup u \quad \text{in } H^1(\Omega)$$
$$u_n \rightharpoonup u \quad \text{in } L^p(\Omega) \quad (1 \leq p < 2^*)$$
$$u_n \rightharpoonup u \quad \text{in } L^q(\Omega, |x|^{-q}) \quad (1 \leq q < 2^*(s))$$
$$u_n \to u \quad \text{a.e. in } \Omega$$

as $n \to \infty$.

For this limit function $u$, we show that $u \not\equiv 0$ a.e. in $\Omega$. Assume that $u \equiv 0$ a.e. in $\Omega$. By the inequality (2) in Lemma 2,

$$\mu_s \left( \int_{\Omega} \frac{|u_n|^{2^*(s)}}{|x|^s} \, dx \right)^{\frac{2}{2^*(s)}} \leq \int_{\Omega} |\nabla u_n|^2 \, dx + C \int_{\Omega} u_n^2 \, dx \quad (8)$$

holds for all $n$. Thus (6), (7), (8) and $u_n \to u$ in $L^2(\Omega)$ yield

$$\mu_s \leq \mu^N_s(\Omega) + o(1).$$

Letting $n$ tend to infinity, we obtain $\mu_s \leq \mu^N_s(\Omega)$ and which is contradiction in the assumption of $\mu^N_s(\Omega) < \mu_s$. Consequently $u \not\equiv 0$.

By the theorem of Brezis and Lieb (see [3]), we obtain

$$\int_{\Omega} \frac{|u_n|^{2^*(s)}}{|x|^s} \, dx = \int_{\Omega} \frac{|u|^{2^*(s)}}{|x|^s} \, dx + \int_{\Omega} \frac{|u_n - u|^{2^*(s)}}{|x|^s} \, dx + o(1)$$

and it follows that

$$1 = \left( \int_{\Omega} \frac{|u|^{2^*(s)}}{|x|^s} \, dx \right)^{\frac{2}{2^*(s)}}$$

$$= \left( \int_{\Omega} \frac{|u|^{2^*(s)}}{|x|^s} \, dx + \int_{\Omega} \frac{|u_n - u|^{2^*(s)}}{|x|^s} \, dx \right)^{\frac{2}{2^*(s)}} + o(1)$$

$$\leq \left( \int_{\Omega} \frac{|u|^{2^*(s)}}{|x|^s} \, dx \right)^{\frac{2}{2^*(s)}} + \left( \int_{\Omega} \frac{|u_n - u|^{2^*(s)}}{|x|^s} \, dx \right)^{\frac{2}{2^*(s)}} + o(1).$$

On the other hand, we have

$$\left( \int_{\Omega} \frac{|u|^{2^*(s)}}{|x|^s} \, dx \right)^{\frac{2}{2^*(s)}} + \left( \int_{\Omega} \frac{|u_n - u|^{2^*(s)}}{|x|^s} \, dx \right)^{\frac{2}{2^*(s)}}$$

$$\leq \frac{1}{\mu_s^N(\Omega)} \int_{\Omega} (|\nabla u| + u^2) \, dx + \frac{1}{\mu_s^N(\Omega)} \int_{\Omega} (|\nabla (u_n - u)|^2 + (u_n - u)^2) \, dx$$

$$= \frac{1}{\mu_s^N(\Omega)} \int_{\Omega} (|\nabla u_n|^2 + u_n^2) \, dx + o(1)$$

$$= 1 + o(1).$$
Hence there exist a limit and we obtain
\[
\lim_{n \to \infty} \left( \int_{\Omega} \frac{|u_n|^{2^*(s)}}{|x|^s} \, dx + \int_{\Omega} \frac{|u_n - u|^{2^*(s)}}{|x|^s} \, dx \right)^{\frac{2}{2^*(s)}} = \lim_{n \to \infty} \left[ \left( \int_{\Omega} \frac{|u|^{2^*(s)}}{|x|^s} \, dx \right)^{\frac{2}{2^*(s)}} + \left( \int_{\Omega} \frac{|u_n - u|^{2^*(s)}}{|x|^s} \, dx \right)^{\frac{2}{2^*(s)}} \right] = 1.
\]

By the equality condition of the above, we get either
\[
u \equiv 0 \quad \text{a.e. in } \Omega \quad \text{or} \quad u_n \to u \neq 0 \quad \text{in } L^{2^*(s)}(\Omega, |x|^{-s}).
\]

Since \(u \neq 0\) we obtain \(u_n \to u \neq 0\) in \(L^{2^*(s)}(\Omega, |x|^{-s})\) and hence this \(u\) is the minimizer of \(\mu_s^N(\Omega)\).

Next, we prove Lemma 3 (II).

Proof (Proof of Lemma 3 (II)) We assume the existence of the minimizer of \(\mu_s^N(r\Omega)\) and derive a contradiction. Let \(u \in H^1(r\Omega)\) be a minimizer of \(\mu_s^N(r\Omega)\), then we have
\[
\mu_s^N(r\Omega) = \int_{r\Omega} (|\nabla u|^2 + u^2) \, dx \geq \int_{r\Omega} (|\nabla u|^2 + \frac{1}{r^2} u^2) \, dx \geq \mu_s^N(r\Omega).
\]

By Lemma 1, the assumption \(\mu_s^N(\Omega) = \mu_s^N(r\Omega) \leq \mu_s\) and \(\mu_s^N(r\Omega) \geq \mu_s^N(r\Omega)\), we have
\[
\mu_s \geq \mu_s^N(r\Omega) > \mu_s^N(r\Omega) = \mu_s^N(\Omega) = \mu_s.
\]

This is a contradiction.

3 Proof of Theorem 1

In this section, we prove Theorem 1.

Proof (Proof of Theorem 1 (I)) We recall that
\[
\mu_s^N(\Omega) := \inf \left\{ \int_{\Omega} (|\nabla u|^2 + u^2) \, dx \mid u \in H^1(\Omega), \int_{\Omega} \frac{|u|^{2^*(s)}}{|x|^s} \, dx = 1 \right\}.
\]

Taking a constant \(C\) such that \(\int_{\Omega} \frac{C^{2^*(s)}}{|x|^s} = 1\) and \(u \equiv C\) as a test function, it follows that
\[
\mu_s^N(\Omega) \leq |\Omega| \left( \int_{\Omega} |x|^{-s} \right)^{-\frac{s}{2^*(s)}}.
\]

If this \(C\) is a minimizer of \(\mu_s^N(\Omega)\), then by Lagrange multiplier theorem \(C\) is a classical solution of
\[
\begin{align*}
-\Delta u + u &= \mu_s^N(\Omega) \frac{u^{2^*(s)}}{|x|^s} \quad \text{in } \Omega \\
\frac{\partial u}{\partial n} &= 0 \quad \text{on } \partial \Omega.
\end{align*}
\]

This contradicts and therefore
\[
\mu_s^N(\Omega) < |\Omega| \left( \int_{\Omega} |x|^{-s} \right)^{-\frac{s}{2^*(s)}}.
\]

Combining this estimate and Lemma 3 (I), Theorem 1 (I) holds true.
Proof (Proof of Theorem 1 (II)) Since Lemma 2, we can define a constant $m$ by
\[
m := \inf \{ C > 0 \mid (2) \text{ holds} \}.
\]
$M$ is defined by $M := \sqrt{m}$. In inequality (2), $C$ is replaced by $M^2$ and hence we have
\[
\mu_s \leq \frac{\int_{\Omega} (|\nabla u|^2 + M^2 u^2)dx}{\left( \int_{\Omega} \frac{|u|^{2^*(s)}}{|x|^{s}} dx \right)^{\frac{s}{2^*(s)}}},
\]
for all $u \in H^1(\Omega)$. Therefore by Lemma 1 we obtain
\[
\mu_s \leq \inf_{u \in H^1(\Omega) \setminus \{0\}} \frac{\int_{\Omega} (|\nabla u|^2 + M^2 u^2)dx}{\left( \int_{\Omega} \frac{|u|^{2^*(s)}}{|x|^{s}} dx \right)^{\frac{s}{2^*(s)}}} = \mu_s^N(M) = \mu_s^N(\Omega).
\]
Recall that $\mu_s^N(\Omega) \leq \mu_s$ holds for all bounded domain $\Omega$ and thus $\mu_s^N(M\Omega) = \mu_s$. Consequently we obtain the result of Theorem 1 (II) by Lemma 3 (II).

4 Singularity on the boundary

Throughout this section, assume that $0 \in \partial \Omega$. If the mean curvature of $\partial \Omega$ at 0 is positive, we have obtained the results in Section 1. However, if the mean curvature of $\partial \Omega$ at 0 vanishes, we don’t obtain results so far, even if the attainability of $\mu_N^s(\Omega)$. In this section, we show the following results by using the strategy in Section 2 and Section 3.

Theorem 3 Let $\Omega \subset \mathbb{R}^N$ is bounded domain with smooth boundary, $0 \in \partial \Omega$ and $\partial \Omega$ is flat near the origin. Then the following statements hold;

(I) If $\Omega$ is sufficiently small, then $\mu_N^s(\Omega)$ is attained. Especially, if $\Omega$ satisfies the following:
\[
|\Omega| \left( \int_{\Omega} |x|^{-s} dx \right)^{-\frac{s}{2^*(s)}} \leq \frac{\mu_s}{2 \pi^{\frac{s}{2}}}.
\]
then $\mu_N^s(\Omega)$ is attained.

(II) There is a positive constant $M$ which depends on only $\Omega$ such that $\mu_N^s(r\Omega)$ is never attained if $r > M$.

This condition of the boundary in this theorem is a special case of vanishing of the mean curvature of $\partial \Omega$ at 0.

We prove the theorem in the same way as in Section 2 and Section 3. Different from the proof of Theorem 1, we need the following lemma instead of Lemma 2.

Lemma 4 There is a positive constant $C$ depends on only $\Omega$ such that
\[
\frac{\mu_s}{2 \pi^{\frac{s}{2}}} \left( \int_{\Omega} \frac{|u|^{2^*(s)}}{|x|^{s}} dx \right)^{\frac{s}{2^*(s)}} \leq \int_{\Omega} |\nabla u|^2 dx + C \int_{\Omega} u^2 dx \quad (u \in H^1(\Omega)).
\]
Proof We introduce some notation. \( B_R(0) \) is an open ball which center is origin and radius is \( R \). \( \mathbb{R}^N_+ \) is a half space which is defined by \( \mathbb{R}^N_+ := \{(x', x_N) \in \mathbb{R}^N | x_N > 0\} \) where \( x' := (x_1, \ldots, x_{N-1}) \in \mathbb{R}^{N-1} \).

Since \( \partial \Omega \) is flat near the origin, by rotating coordinate there is a constant \( r > 0 \) such that \( B_r(0) \cap \Omega = B_r^+(0) := B_r(0) \cap \mathbb{R}^N_+ \). For \( u \in H^1(\Omega) \) we have

\[
\left( \int_{\Omega} \frac{|u|^{2^*(s)}}{|x|^s} \, dx \right)^{\frac{2}{2^*(s)}} = \left( \int_{B_r^+(0)} \frac{|u|^{2^*(s)}}{|x|^s} \, dx + \int_{\Omega \setminus B_r^+(0)} \frac{|u|^{2^*(s)}}{|x|^s} \, dx \right)^{\frac{2}{2^*(s)}} \\
\leq \left( \int_{B_r^+(0)} \frac{|u|^{2^*(s)}}{|x|^s} \, dx \right)^{\frac{2}{2^*(s)}} + \left( \int_{\Omega \setminus B_r^+(0)} \frac{|u|^{2^*(s)}}{|x|^s} \, dx \right)^{\frac{2}{2^*(s)}} = J_1 + J_2.
\]

For \( u \in H^1(B_r^+(0)), \tilde{u} \in H^1(B_r(0)) \) is defined by the even reflection for the direction \( x_N \), that is,

\[
\tilde{u}(x', x_N) := \begin{cases} 
  u(x', x_N) & \text{if } 0 \leq x_N < 1 \\
  u(x', x_N) & \text{if } -1 < x_N < 0.
\end{cases}
\]

Concerning \( J_1 \), by Lemma 2 we have

\[
J_1 = \left( \int_{B_r^+(0)} \frac{|u|^{2^*(s)}}{|x|^s} \, dx \right)^{\frac{2}{2^*(s)}} \\
= \left( \frac{1}{2} \right)^{\frac{2}{2^*(s)}} \mu_s^{-1} \left( \int_{B_r(0)} |\nabla \tilde{u}|^2 \, dx + C_1 \int_{B_r(0)} \tilde{u}^2 \, dx \right) \\
\leq \left( \frac{1}{2} \right)^{\frac{2}{2^*(s)}} \mu_s^{-1} \cdot 2 \left( \int_{B_r^+(0)} |\nabla u|^2 \, dx + C_1 \int_{B_r^+(0)} u^2 \, dx \right) \\
= \left( \frac{\mu_s}{2 \mu_s - 1} \right)^{-1} \left( \int_{B_r^+(0)} |\nabla u|^2 \, dx + C_1 \int_{B_r^+(0)} u^2 \, dx \right)
\]

for some positive constant \( C_1 \) depends on only \( B_r(0) \).

Next, we estimate \( J_2 \). Let \( \delta > 0 \) for sufficiently small. We consider \( \{\phi_i\}_{i=1}^m \) a partition of unity on \( \Omega \setminus B_r^+(0) \) such that \( \phi_i^{\frac{1}{2}} \in C^1 \) and \( |\text{supp } \phi_i| \leq \delta \) for all \( i \). Since \( |x|^{-s} \leq r^{-s} \) for \( x \in \Omega \setminus B_r^+(0) \) we have

\[
J_2 = \left( \int_{\Omega \setminus B_r^+(0)} \frac{|u|^{2^*(s)}}{|x|^s} \, dx \right)^{\frac{2}{2^*(s)}} \\
\leq \sum_{i=1}^m \left( \int_{\Omega \setminus B_r^+(0)} \frac{\phi_i^{\frac{1}{2}} u^{2^*(s)}}{|x|^s} \, dx \right)^{\frac{2}{2^*(s)}} \\
\leq r^{-\frac{2s}{2^*(s)}} \sum_{i=1}^m \left( \int_{\Omega \setminus B_r^+(0)} \frac{\phi_i^{\frac{1}{2}} u^{2^*(s)}}{|x|^s} \, dx \right)^{\frac{2}{2^*(s)}}.
\]
By Hölder inequalities it follows that

\[
\left( \int_{\Omega \setminus B^+_i(0)} |\phi_i^{\frac{1}{2}} u|^{2^*(s)} \, dx \right)^{\frac{2}{2^*(s)}} \leq |\text{supp} \phi_i|^{\frac{2}{2^*(s)} - \frac{2}{s}} \|\phi_i^{\frac{1}{2}} u\|_{L^{2^*(s)}(\Omega \setminus B^+_i(0))}^2 \\
\leq \delta^{\frac{2}{2^*(s)} - \frac{2}{s}} \|\phi_i^{\frac{1}{2}} u\|_{L^{2^*(s)}(\Omega \setminus B^+_i(0))}^2
\]

for each \( i \in \mathbb{N} \). Since \( \delta \) is sufficiently small, by using the Sobolev inequalities (If necessary we use the Sobolev inequalities of mixed boundary condition version.) we have

\[
J_2 \leq \left( \frac{\mu_s}{2^{\frac{4}{N}}} \right)^{-1} \frac{1}{2} \sum_{i=1}^{m} \int_{\Omega \setminus B^+_i(0)} |\nabla (\phi_i^{\frac{1}{2}} u)|^2 \, dx.
\]

Consequently we have

\[
J_2 \leq \left( \frac{\mu_s}{2^{\frac{4}{N}}} \right)^{-1} \left( \int_{\Omega \setminus B^+_i(0)} |\nabla u|^2 \, dx + C_2 \int_{\Omega \setminus B^+_i(0)} u^2 \, dx \right).
\]

for some positive constant \( C_2 \) depends on only \( \Omega \setminus B^+_i(\Omega) \). Combining the estimates of \( J_1 \) and \( J_2 \) we obtain

\[
\left( \int_{\Omega} \frac{|u|^{2^*(s)}}{|x|^q} \right)^{\frac{2}{2^*(s)}} \leq J_1 + J_2 \leq \left( \frac{\mu_s}{2^{\frac{4}{N}}} \right)^{-1} \left( \int_{\Omega} |\nabla u|^2 \, dx + C \int_{\Omega} u^2 \, dx \right)
\]

for some positive constant \( C \) depends on \( \Omega \).

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**References**