SINGULAR EXTREMAL SOLUTIONS TO A LIOUVILLE-GELFAND TYPE PROBLEM WITH EXPONENTIAL NONLINEARITY

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SINGULAR EXTREMAL SOLUTIONS TO A LIOUVILLE-GELFAND TYPE PROBLEM WITH EXPONENTIAL NONLINEARITY

FUTOSHI TAKAHASHI

Abstract. We consider a Liouville-Gelfand type problem

\[- \Delta u = e^u + \lambda f(x) \quad \text{in } \Omega, \quad u > 0 \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial \Omega,\]

where \( \Omega \subset \mathbb{R}^N \) \((N \geq 1)\) is a smooth bounded domain, \( f \geq 0, \)
\( f \not\equiv 0 \) is a given smooth function, and \( \lambda \geq 0 \) is a parameter. We
are concerned with the regularity property of extremal solutions
to the problem, and prove that there exists a domain \( \Omega \) and a
smooth nonnegative function \( f \) such that the extremal solution of
the problem is singular when the dimension \( N \geq 10 \). This result
is sharp in the sense that the extremal solution is always regular
(bounded) for any \( f \) and \( \Omega \) when \( 1 \leq N \leq 9 \).

1. Introduction.

In this paper, we consider a Liouville-Gelfand type problem with the
exponential nonlinearity:

\[
\begin{cases}
-\Delta u = e^u + \lambda f(x) & \text{in } \Omega, \\
u > 0 & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]

(1.1)

where \( \Omega \subset \mathbb{R}^N \), \( N \geq 1 \) is a smooth bounded domain, \( f \in C^\infty(\Omega) \) is
a nonnegative function, not identically equal to zero, and \( \lambda \geq 0 \) is a
parameter.

First, we recall the notion of a weak solution to (1.1); see Brezis et
al. [2].

Definition 1.1. A function \( u \in L^1(\Omega) \) is called a weak solution to
(1.1) if \( u > 0 \) in \( \Omega \), \( e^u \delta \in L^1(\Omega) \), and

\[
- \int_{\Omega} u \Delta \zeta \, dx = \int_{\Omega} (e^u + \lambda f) \zeta \, dx
\]

(1.2)
holds for any $\zeta \in C^2(\Omega)$ such that $\zeta = 0$ on $\partial \Omega$, where $\delta(x) = \text{dist}(x, \partial \Omega)$.

Note that since $|\zeta| \leq C\delta$ for any $\zeta \in C^2(\Omega)$, $\zeta = 0$ on $\partial \Omega$, the integral of the right hand side of (1.2) is well-defined.

By the methods in [2], [3] and [8], we can prove the following basic facts concerning the problem (1.1)$_{\lambda}$.

**Proposition 1.2.** Let $f \in C^\infty(\Omega)$, $f \geq 0, f \neq 0$ be a given function. Then there exists $\lambda^* \in (0, +\infty)$, called an extremal parameter, such that the followings hold true.

(i) For $\lambda \in (0, \lambda^*)$, there exists a minimal solution $u_{\lambda}$ to (1.1)$_{\lambda}$. $u_{\lambda}$ is smooth, stable in the sense that

\[(1.3) \quad \int_{\Omega} |\nabla \phi|^2 dx \geq \int_{\Omega} e^{u_{\lambda}} \phi^2 dx\]

holds for any $\phi \in C^1_0(\Omega)$. Furthermore, $u_{\lambda}$ depends continuously and monotone increasingly on $\lambda \in (0, \lambda^*)$.

(ii) For $\lambda = \lambda^*$, there exists a unique weak solution $u^*$ to (1.1)$_{\lambda}$. $u^*$ is called the extremal solution and is obtained as an increasing limit of the minimal solutions $u_{\lambda}$:

$$u^*(x) = \lim_{\lambda \uparrow \lambda^*} u_{\lambda}(x) \quad (x \in \Omega).$$

(iii) For $\lambda > \lambda^*$, there is no solution to (1.1)$_{\lambda}$, even in the weak sense.

In this paper, we concern the regularity issue of the extremal solution $u^*$ in Proposition 1.2 (ii). In some cases, $u^*$ may be singular (i.e., $u^* \not\in L^\infty(\Omega)$), but little is known about the singular extremal solutions.

For the well-studied problem

\[(1.4) \quad \begin{cases} -\Delta u = \lambda e^u & \text{in } \Omega, \\ u = 0 & \text{on } \partial \Omega, \end{cases}\]

we have also the extremal parameter $\lambda^* \in (0, +\infty)$ for which there is a minimal, strict stable solution for $0 < \lambda < \lambda^*$, the unique extremal solution (may be singular) for $\lambda = \lambda^*$, and no solution for $\lambda > \lambda^*$ even in the weak sense [2] [8]. If $\Omega = B$, the unit ball in $\mathbb{R}^N$, and $N \geq 10$, then the explicit radial function $v(x) = -2 \log |x|$ becomes the singular extremal solution of (1.4) for $\lambda = 2(N-2)$ [3]. Note that $v \in H^1_0(B)$ if $N \geq 3$. On the other hand, the extremal solution of (1.4) is bounded on any bounded smooth domain $\Omega$ when $1 \leq N \leq 9$ [4], [9]. The readers are recommended to refer to the recent book by Dupaigne [7] and its references for these results. Concerning the
existence of singular solutions. Dávila and Dupaigne [6] prove that there exists an 1-parameter family of singular solutions \((u(t), \lambda(t))_{t > 0}\) to (1.4) for \(\lambda = \lambda(t)\) with the property
\[
\|u(t) - \log \frac{1}{|\cdot - \xi(t)|^2}\|_{L^\infty(\Omega)} + |\lambda(t) - 2(N - 2)| \to 0 \quad (t \to 0)
\]
for some \(\xi(t) \in \Omega\), where the domain \(\Omega\) is a small perturbation of a ball in an appropriate sense in \(\mathbb{R}^N, N \geq 4\). The authors also prove that these singular solutions correspond to the extremal solutions when \(N \geq 11\). Recently, Miyamoto [10] studies the perturbed Liouville-Gelfand problem on the unit ball \(B\) in \(\mathbb{R}^N, N \geq 3\):
\[
\begin{cases}
-\Delta u = \lambda(e^u + g(u)) & \text{in } B, \\
u > 0 & \text{in } B, \\
u = 0 & \text{on } \partial B,
\end{cases}
\]
where \(g \in C^1\) is an appropriate nonlinearity which is “small” compared to \(e^u\). The author proves the existence of radial singular solution \((u^*, \lambda^*)\) with the property
\[
u^*(|x|) \sim -2 \log |x| - \log \lambda^* + \log 2(N - 2) \quad (|x| \to 0),
\]
and if \(N \geq 10\), this singular radial solution corresponds to the extremal solution.

For other nonlinearities, Dávila [5] studies the regularity and singularity issue of extremal solutions to the problem
\[
\begin{cases}
-\Delta u = u^p + \lambda f(x) & \text{in } \Omega, \\
u > 0 & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega
\end{cases}
\]
where \(\Omega \subset \mathbb{R}^N, N \geq 1\) is a smooth bounded domain, \(f \in C^\infty(\Omega)\) is a nonnegative function, not identically equal to zero, and \(\lambda > 0\). The results in this paper correspond to the ones in [5] for the exponential nonlinear case.

This paper is organized as follows: In §2, we prove that the extremal solutions are regular for any \(f\) and \(\Omega\) when \(1 \leq N \leq 9\). In §3, we examine the sharpness of this regularity theorem in terms of the dimension of the domain, and prove that there exists a bounded domain \(\Omega\) and a smooth \(f \geq 0, f \not\equiv 0\) such that the extremal solution \(u^*\) is not bounded when \(N \geq 10\). This means that the assumption \(1 \leq N \leq 9\) in the regularity theorem in §2 is sharp and cannot be relaxed in general. Finally in §4, we treat the case when the domain is a ball.
2. Extremal solutions are regular for $1 \leq N \leq 9$.

First, we prove the boundedness of the extremal solution to (1.1) in lower dimensions.

**Theorem 2.1.** Let $\Omega$ be any smooth bounded domain in $\mathbb{R}^N$ and let $f \in C^\infty(\Omega)$, $f \geq 0$, $f \not\equiv 0$ be any given function. If $1 \leq N \leq 9$, then there exists a constant $C > 0$ such that for any $0 < \lambda < \lambda^*$, it holds

$$\|u_\lambda\|_{L^\infty(\Omega)} \leq C$$

for the minimal solution $u_\lambda$ to (1.1). Consequently, the extremal solution $u^*$ is bounded, hence smooth.

**Proof.** We follow the arguments in [4], [9] with some modifications for our context. Recall the minimal solution $u = u_\lambda$ satisfies the stability inequality

$$\int_\Omega |\nabla \phi|^2 dx \geq \int_\Omega e^u \phi^2 dx, \quad \forall \phi \in C^1_0(\Omega)$$

and the weak form of the equation

$$\int_\Omega \nabla \psi \cdot \nabla u dx = \int_\Omega (e^u + \lambda f) \psi dx, \quad \forall \psi \in C^1_0(\Omega).$$

We put $\phi = e^{tu} - 1$ and $\psi = \frac{t}{2} (e^{2tu} - 1)$, where $t > 0$. Testing with them, we have

$$\int_\Omega t^2 e^{2tu} |\nabla u|^2 dx \geq \int_\Omega e^u (e^{tu} - 1)^2 dx$$

and

$$\int_\Omega t^2 e^{2tu} |\nabla u|^2 dx = \frac{t}{2} \int_\Omega (e^u + \lambda f) (e^{2tu} - 1) dx.$$ Combining these, we obtain

$$\int_\Omega e^u (e^{tu} - 1)^2 dx \leq \frac{t}{2} \int_\Omega (e^u + \lambda f) (e^{2tu} - 1) dx,$$

which in turn implies

$$\left(1 - \frac{t}{2}\right) \int_\Omega e^{(2t+1)u} dx \leq \int_\Omega \left(2e^{(t+1)u} - \left(\frac{t}{2} + 1\right) e^u + \frac{t}{2} (e^{2tu} - 1) f\right) dx \leq 2 \int_\Omega e^{(t+1)u} dx + \frac{\lambda t}{2} \int_\Omega e^{2tu} f dx \leq 2 \left(\int_\Omega e^{(2t+1)u} dx\right)^{\frac{t+1}{2t+1}} |\Omega|^{\frac{1}{2t+1}} + \frac{t\lambda^*}{2} \left(\int_\Omega e^{(2t+1)u} dx\right)^{\frac{2t}{2t+1}} \left(\int_\Omega f^{2t+1} dx\right)^{\frac{1}{2t+1}}.$$
We may assume that
\[ \int_{\Omega} e^{(2t+1)u} dx > 1, \]
because on the contrary, we have \( \|e^u\|_{L^{2t+1}(\Omega)} \leq 1 \), and the estimate is independent of \( \lambda \in (0, \lambda^*) \). In this case, if \( 1 - \frac{t}{2} > 0 \) and \( \frac{t+1}{2t+1} < \frac{2t}{2t+1} \), that is, if \( 1 < t < 2 \), then we have
\[ \int_{\Omega} e^{(2t+1)u} dx \leq \left[ \left( 1 - \frac{t}{2} \right)^{-1} \left\{ 2|\Omega|^\frac{1}{2t+1} + \frac{t\lambda^*}{2} \left( \int_{\Omega} f^{2t+1} dx \right)^\frac{1}{2t+1} \right\} \right]^{2t+1} =: C, \]
here \( C = C(|\Omega|, f) \) is independent of \( \lambda \in (0, \lambda^*) \). Thus we have \( \|e^u\|_{L^{2t+1}(\Omega)} \leq C \), which implies
\[ \|e^{\lambda u} + \lambda f\|_{L^{2t+1}(\Omega)} \leq C \]
when \( 1 < t < 2 \). Now, standard elliptic estimates and Sobolev embedding imply that \( \|u_\lambda\|_{L^\infty(\Omega)} \leq C \) uniformly in \( \lambda \) if \( 2(2t + 1) > N \). Since we may choose \( t \in (1, 2) \) very close to 2, we obtain the uniform \( L^\infty \) bound for \( u_\lambda \) when \( N \leq 9 \). This proves Theorem 2.1.

3. Singular extremal solutions when \( N \geq 10 \).

In this section, we prove the following theorem, which says that the restriction of the dimension in Theorem 2.1 is sharp concerning the boundedness of the extremal solutions.

**Theorem 3.1.** Let \( \Omega \) be a smooth bounded domain in \( \mathbb{R}^N \). Assume that \( N \geq 10 \), \( 0 \in \Omega \) and
\[ (3.1) \quad \max_{x \in \partial \Omega} |x|^2 \leq 2(N - 2) \]
holds true. Then there exists \( f \in C^\infty(\Omega) \), \( f \geq 0 \), \( f \neq 0 \) such that the extremal solution \( u^* \) to (1.1) with \( f \) satisfies
\[ u^* \notin L^\infty(\Omega) \quad \text{and} \quad \lambda^* = 1. \]

In the proof of Theorem 3.1, we need a characterization of the unbounded extremal solutions in the energy class \( H^1(\Omega) \), which is similar to Brezis and Vázquez [3], Theorem 3.1. See also Dávila [5], Lemma 4.

**Lemma 3.2.** Let \( u \in H^1_0(\Omega) \), \( u \notin L^\infty(\Omega) \), be a singular weak solution to (1.1)_\lambda. Then the followings are equivalent:
(i) $e^u \delta \in L^1(\Omega)$ and
\[
\int_{\Omega} |\nabla \phi|^2 dx \geq \int_{\Omega} e^u \phi^2 dx
\]
holds for every $\phi \in C_0^1(\Omega)$.

(ii) $\lambda = \lambda^*$ and $u = u^*$.

Proof. The implication $(ii) \implies (i)$ follows easily by the stability property of the minimal solutions $u_\lambda$ and Fatou’s lemma.

Let us prove $(i) \implies (ii)$. Since no solution exists for $\lambda > \lambda^*$ by Proposition 1.2, we have $\lambda \leq \lambda^*$. Assume the contrary that $\lambda < \lambda^*$.

By the density argument and the fact that $u, u_\lambda \in H^1_0(\Omega)$, we can take the test function $\phi = u - u_\lambda \in H^1_0(\Omega)$. By the minimality of $u_\lambda$, we see $u - u_\lambda \geq 0$ in $\Omega$, and the assumption $u \not\in L^\infty(\Omega)$ implies that $u - u_\lambda \not\equiv 0$, since $u_\lambda$ is bounded for $\lambda < \lambda^*$. Combining the equation satisfied by $u - u_\lambda$ with $(i)$, we obtain
\[
\int_{\Omega} (e^u + \lambda f - e^{u_\lambda} - \lambda f)(u - u_\lambda) dx = \int_{\Omega} |\nabla (u - u_\lambda)|^2 dx \geq \int_{\Omega} e^u (u - u_\lambda)^2 dx,
\]
which implies
\[
\int_{\Omega} (u - u_\lambda)(e^u - e^{u_\lambda} - e^u(u - u_\lambda)) dx \geq 0.
\]
Since the integrand is non positive by the convexity of $s \mapsto e^s$, we conclude that $e^u = e^{u_\lambda} + e^u(u - u_\lambda)$ a.e. on $\Omega$. Again the strict convexity of $s \mapsto e^s$ implies $u = u_\lambda$ a.e. on $\Omega$, which is a contradiction. Thus we must have $\lambda = \lambda^*$.

In the following, let $v_s$ denote the explicit singular radial function defined as
\[v_s(x) = -2 \log |x| + \log 2(N - 2), \quad x \in \mathbb{R}^N.\]
Then $v_s \in H^1_{\text{loc}}(\mathbb{R}^N)$ if $N \geq 3$ and $v_s$ satisfies the equation $-\Delta v = e^v$ in $\mathbb{R}^N$. Recall we have assumed $0 \in \Omega$ in Theorem 3.1. As in [5], our strategy is to look for a singular solution $u$ to (1.1) (with a suitable $f$) of the form
\[u = v_s - \psi\]
for some $\psi \in C^\infty(\Omega), \psi \geq 0$. The extremality of $u$ will follow from the fact that $u \in H^1_0(\Omega)$ and Lemma 3.2.

Next simple lemma is well-known and in fact is used in [5].
Lemma 3.3. Let $\Omega$ be a smooth bounded domain in $\mathbb{R}^N$ and $\omega$ be a smooth subdomain of $\Omega$ with $\overline{\omega} \subset \Omega$. Let $\psi$ satisfy
\[
\begin{cases}
\Delta \psi = 0 & \text{in } \Omega \setminus \overline{\omega}, \\
\psi = 0 & \text{on } \partial \omega, \\
\frac{\partial \psi}{\partial \nu} \geq 0 & \text{on } \partial \omega,
\end{cases}
\]
where $\nu$ is the unit normal vector on $\partial \omega$ pointing to the inside of $\Omega \setminus \overline{\omega}$. Then if we put
\[
\begin{cases}
\overline{\psi} = \psi & \text{on } \Omega \setminus \overline{\omega}, \\
\overline{\psi} = 0 & \text{on } \omega,
\end{cases}
\]
$\overline{\psi}$ satisfies
\[\Delta \overline{\psi} \geq 0 \quad \text{in } D'(\Omega).\]

Proof. For any $\phi \in D(\Omega)$, $\phi \geq 0$, we have
\[
\int_{\Omega} \overline{\psi} \Delta \phi dx = \int_{\Omega \setminus \overline{\omega}} \psi \Delta \phi dx = \int_{\Omega \setminus \overline{\omega}} \phi \Delta \psi dx \\
+ \int_{\partial(\Omega \setminus \overline{\omega})} \frac{\partial \phi}{\partial \nu} \psi dx - \int_{\partial(\Omega \setminus \overline{\omega})} \frac{\partial \psi}{\partial \nu} \phi dx.
\]
Now,
\[
\int_{\partial(\Omega \setminus \overline{\omega})} \frac{\partial \phi}{\partial \nu} \psi dx = \int_{\partial \Omega} \frac{\partial \phi}{\partial \nu} \psi dx - \int_{\partial \omega} \frac{\partial \phi}{\partial \nu} \psi dx = 0
\]
since $\psi = 0$ on $\partial \omega$ and $\frac{\partial \phi}{\partial \nu} = 0$ on $\partial \Omega$. On the other hand,
\[
- \int_{\partial(\Omega \setminus \overline{\omega})} \frac{\partial \psi}{\partial \nu} \phi dx = - \left( \int_{\partial \Omega} \frac{\partial \psi}{\partial \nu} \phi dx - \int_{\partial \omega} \frac{\partial \psi}{\partial \nu} \phi dx \right) = \int_{\partial \omega} \frac{\partial \psi}{\partial \nu} \phi dx \geq 0
\]
by $\frac{\partial \psi}{\partial \nu} \geq 0$ and $\phi \geq 0$. Thus we obtain
\[
\int_{\Omega} \overline{\psi} \Delta \phi dx = - \int_{\partial(\Omega \setminus \overline{\omega})} \frac{\partial \psi}{\partial \nu} \phi dx \geq 0,
\]
which proves the lemma. \hfill \Box

Next is a variant of [5]: Lemma 5.

Lemma 3.4. Let $\Omega$ be a smooth bounded domain in $\mathbb{R}^N$, $0 \in \Omega$, satisfying the assumption (3.1). Then there exists a function $\psi \in C^\infty(\overline{\Omega})$ such that
\[
\begin{align*}
(i) \quad & \psi \geq 0 \quad \text{in } \overline{\Omega}, \\
(ii) \quad & \Delta \psi \geq 0 \quad \text{in } \Omega, \\
(iii) \quad & \psi \equiv 0 \quad \text{in a neighborhood of } 0 \in \Omega,
\end{align*}
\]
Proof. This lemma is essentially the same one in Dávila [5]. We recall the proof here for the reader’s convenience.

Put \( r = \frac{1}{2} \text{dist}(0, \partial \Omega) \) and let \( B_r \) denote the open ball with center 0 and radius \( r \). Note that the smallness assumption of \( \Omega \) (3.1) implies that \( v_s(x) \geq 0 \) for \( x \in \partial \Omega \). Now, let \( \psi_1 \) be the solution of

\[
\begin{cases}
\Delta \psi_1 = 0 & \text{in } \Omega \setminus \overline{B}_r, \\
\psi_1 = v_s & \text{on } \partial \Omega, \\
\psi_1 = 0 & \text{on } \partial B_r
\end{cases}
\]

where \( v_s \) is defined in (3.2). Then \( \psi_1 \) is smooth and by the maximum principle, \( \psi_1 > 0 \) on \( \Omega \setminus \overline{B}_r \). Thus \( \frac{\partial \psi_1}{\partial \nu} > 0 \) by the Hopf lemma, where \( \nu \) is the unit normal vector on \( \partial B_r \) pointing to the inside of \( \Omega \setminus \overline{B}_r \). Put

\[
\begin{cases}
\bar{\psi}_1 = \psi_1 & \text{on } \Omega \setminus \overline{B}_r, \\
\bar{\psi}_1 = 0 & \text{on } B_r.
\end{cases}
\]

Then by Lemma 3.3, we have

\[ \Delta \bar{\psi}_1 \geq 0 \quad \text{in } D'(\Omega). \]

Put

\[ \psi = \bar{\psi}_1 * \rho_\varepsilon \]

where \( \rho_\varepsilon(x) = \frac{1}{\varepsilon^N} \rho(\frac{x}{\varepsilon}) \) with \( \rho \) satisfying \( \rho \in C_0^\infty(\mathbb{R}^N) \), \( \rho \geq 0 \), \( \rho(x) = \rho(|x|) \), \( \text{supp} (\rho) \subset B_1 \), and \( \int_{\mathbb{R}^N} \rho dx = 1 \). Then we check that \( \psi \) is the desired function. \( \square \)

Proof of Theorem 3.1. Let \( u = v_s - \psi \), where \( v_s \) is an explicit singular solution (3.2) and \( \psi \in C^\infty(\overline{\Omega}) \) is as in Lemma 3.4. Since we assume \( 0 \in \Omega \), we have \( u \not\in L^\infty(\Omega) \). By Lemma 3.4 (ii) and (iv), we have

\[ -\Delta u = -\Delta v_s + \Delta \psi = e^{v_s} + \Delta \psi \geq e^{v_s} > 0 \]

on \( \Omega \) and \( u = 0 \) on \( \partial \Omega \). Thus \( u \geq 0 \) by the maximum principle. Now, put

\[ f(x) = e^{v_s} + \Delta \psi - e^u = e^{v_s} - e^{v_s-\psi} + \Delta \psi. \]

Then \( f \geq 0 \) in \( \overline{\Omega} \) since \( v_s \geq u \) by Lemma 3.4 (i) and (ii). Also, we have

\[ -\Delta u = e^{v_s} + \Delta \psi = e^u + f(x) \]

in \( \Omega \). Furthermore, by Lemma 3.4 (iv),

\[ f(x) = e^{v_s(x)}(1 - e^{-\psi(x)}) + \Delta \psi(x) = \Delta \psi(x) \]

for \( x \) in a neighborhood of 0. Thus \( f \) is smooth on \( \Omega \).
Finally, we check that \( u \) is stable in the sense of (1.3). Indeed, for any \( \phi \in C^1_0(\Omega) \), we have
\[
\int_{\Omega} e^u \phi^2 \, dx \leq \int_{\Omega} e^{v_s} \phi^2 \, dx = 2(N - 2) \int_{\Omega} \frac{\phi^2}{|x|^2} \, dx \\
\leq 2(N - 2) \left( \frac{2}{N - 2} \right)^2 \int_{\Omega} |\nabla \phi|^2 \, dx \\
\leq \int_{\Omega} |\nabla \phi|^2 \, dx,
\]
here we have used the fact \( u \leq v_s \) for the first inequality, the Hardy inequality
\[
\left( \frac{N - 2}{2} \right)^2 \int_{\Omega} \frac{\phi^2}{|x|^2} \, dx \leq \int_{\Omega} |\nabla \phi|^2 \, dx \quad \forall \phi \in C^1_0(\Omega)
\]
for the second inequality. Note that the assumption \( N \geq 10 \) is equivalent to \( 2(N - 2) \left( \frac{2}{N - 2} \right)^2 \leq 1 \) for the third inequality.

Thus \( u \) is an unbounded, stable, \( H^1_0 \)-solution of (1.1) (with \( \lambda = 1 \)). By the characterization of the singular energy extremal solutions Lemma 3.2, we conclude that \( u = u^* \) and \( \lambda^* = 1 \).

4. THE BALL CASE.

In this section, we treat the case where the domain is a ball. Note that in this case, the minimal solution \( u_\lambda \) of (1.1)_\lambda is radially symmetric if \( f \) is assumed to be radial. More generally, we prove the lemma below, which is a slight modification of Proposition 1.3.4 in [7].

**Lemma 4.1.** Let \( g \in C^1(\mathbb{R}) \). Let \( \Omega \) be a smooth bounded, radially symmetric domain with the symmetric center the origin (ball or annulus) in \( \mathbb{R}^N, N \geq 2 \), and \( f = f(x) \) be a smooth radial function. If \( u \in C^2(\Omega) \) is a stable solution of
\[
\begin{cases}
-\Delta u = g(u) + f(x) & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]
then \( u \) is radially symmetric.

**Proof.** We show that any tangential derivative \( h = x_i u_{x_j} - x_j u_{x_i}, (i, j \in \{1, \cdots, N\}) \) must satisfy \( h \equiv 0 \). First, by integrating by parts and using the boundary condition, we have
\[
\int_{\Omega} h \, dx = \int_{\partial \Omega} (x_i \nu_j - x_j \nu_i) u \, ds_x = 0,
\]
here $\nu_i$ denotes the $i$-th component of the unit normal vector $\nu$ to $\partial \Omega$. Next, by differentiating the equation, we have

$$-\Delta h = g'(u)h + x_i f_{x_j} - x_j f_{x_i} = g'(u)h \quad \text{in } \Omega$$

since $\Delta(x_i u_{x_j}) = x_i \Delta u_{x_j} + 2u_{x_i x_j}$ and $f$ is radially symmetric. Also we have $h = 0$ on $\partial \Omega$ since $\nabla u \perp \partial \Omega$ and thus $x \wedge \nabla u = 0$ on $\partial \Omega$, where $\wedge$ denotes the exterior product. Then, multiplying $h$ and integrating by parts, we obtain

$$\int_{\Omega} |\nabla h|^2 dx - \int_{\Omega} g'(u)h^2 dx = 0.$$

Since $u$ is stable, this means that $h$ is a minimizer of

$$\inf_{\phi \in H_0^1(\Omega), \phi \neq 0} \frac{\int_{\Omega} |\nabla \phi|^2 dx - \int_{\Omega} g'(u)\phi^2 dx}{\int_{\Omega} \phi^2 dx}$$

if $h \neq 0$. Thus the linearized operator $-\Delta - g'(u)\cdot$ (acting on $H_0^1(\Omega)$) has the smallest eigenvalue $\lambda_1(-\Delta - g'(u)\cdot) = 0$, and $h \neq 0$ is the first eigenfunction corresponding to $\lambda_1(-\Delta - g'(u)\cdot)$. But in this case, $h$ must be of constant sign on $\Omega$, which contradicts the fact $\int_{\Omega} h dx = 0$. Thus we obtain $h \equiv 0$, which in turn implies $u$ is radial.

If the domain is a ball, we obtain the following result.

**Theorem 4.2.** Let $B$ denote the open unit ball in $\mathbb{R}^N$ and assume that $f \geq 0$, $f \not\equiv 0$ be any smooth radially symmetric function. If $N \geq 10$, then the extremal solution $u^*$ of the problem

$$\begin{cases}
-\Delta u = e^u + \lambda f & \text{in } B, \\
u = 0 & \text{on } \partial B
\end{cases}$$

satisfies $u^* \notin L^\infty(B)$.

**Proof.** First, we recall the improved Hardy inequality by Brezis and Vázquez [3]: For any bounded domain $\Omega \subset \mathbb{R}^N$, $N \geq 2$, and for any $\phi \in H_0^1(\Omega)$, it holds that

$$\int_{\Omega} |\nabla \phi|^2 dx \geq \left( \frac{N-2}{2} \right)^2 \int_{\Omega} \frac{\phi^2}{|x|^2} dx + H_2 \left( \frac{\omega_N}{|\Omega|} \right)^{\frac{2}{N}} \int_{\Omega} \phi^2 dx,$$

where $H_2$ is the first Dirichlet eigenvalue of the Laplacian on the unit ball in $\mathbb{R}^2$ and $\omega_N$ is the measure of the unit ball in $\mathbb{R}^N$. By this inequality, we derive that the linearized operator $-\Delta - e^{v_*} \cdot = -\Delta - \frac{e^{v_*}}{|x|^2}$, (acting on $H_0^1(\Omega)$), where $v_*$ is a function as in (3.2), has a strict positive first eigenvalue. This fact in turn implies that the maximum principle is valid for the operator $-\Delta - e^{v_*} \cdot$; see, for example, [1].
Next, we claim that \( u_\lambda < v_s \) holds for the minimal solution \( u_\lambda \) for any \( \lambda \in (0, \lambda^*) \). Indeed, \( u_\lambda \) is radial by Lemma 4.1. Assume the contrary that there exists \( r \in (0, 1) \) such that \( u_\lambda(r) \geq v_s(r) \) for some \( \lambda \in (0, \lambda^*) \), where \( r = |x| \). Then \( u_\lambda - v_s \geq 0 \) on \( \partial B_r \) and
\[
-\Delta (u_\lambda - v_s) = e^{u_\lambda} - e^{v_s} + \lambda f \geq e^{v_s}(u_\lambda - v_s) + \lambda f
\]
by the convexity of \( s \mapsto e^s \). Thus
\[
-\Delta (u_\lambda - v_s) - e^{v_s}(u_\lambda - v_s) \geq 0 \quad \text{on } B_r
\]
and we have \( u_\lambda - v_s \geq 0 \) on \( B_r \) by the maximum principle for the operator \( -\Delta - e^{v_s} \). But this is impossible since \( 0 \in B_r \), \( u_\lambda \in L^\infty(B_r) \) and \( v_s \not\in L^\infty(B_r) \). Thus we obtain the claim. By letting \( \lambda \to \lambda^* \), we also get that \( u^* \leq v_s \) on \( B_r \).

By the above claim, we obtain that
\[
\int_B |\nabla \phi|^2 dx - \int_B e^{u^*} \phi^2 dx \geq \inf_{\|\phi\|_{L^2(B)} = 1} \left\{ \int_B |\nabla \phi|^2 dx - \int_B e^{v_s} \phi^2 dx \right\}
\]
for any \( \phi \in H^1_0(B) \) with \( \|\phi\|_{L^2(B)} = 1 \). The right hand side is strictly positive by the improved Hardy inequality and the assumption \( N \geq 10 \). On the other hand, if \( u^* \) is the classical solution to (1.1)\( _{\lambda^*} \), the first eigenvalue of the operator \( -\Delta - e^{u^*} \) (acting on \( H^1_0(B) \))
\[
\lambda_1(-\Delta - e^{u^*}) = \inf_{\phi \in H^1_0(B), \phi \neq 0} \frac{\int_B |\nabla \phi|^2 dx - \int_B e^{u^*} \phi^2 dx}{\int_B \phi^2 dx}
\]
must be 0 by the Implicit Function Theorem. Thus \( u^* \) cannot be bounded. This proves Theorem 4.2.

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