

Strategy-Proof and Non-Exclusive Mechanisms in Probabilistic Provision of an Excludable Public Good

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Abstract

We study strategy-proof probabilistic mechanisms in a binary excludable public good model. We define a new probabilistic mechanism, called the non-exclusive mechanism. We first show that the non-exclusive mechanism satisfies strategy-proofness. Next, we establish that the non-exclusive mechanism is second-best efficient. Finally, we calculate the supremal welfare loss of the non-exclusive mechanism, and compare it with that of the mechanisms on the previous research.

Keywords: Strategy-proofness; Probabilistic mechanism; Excludable public good; Second-best efficiency; Supremal welfare loss.

JEL codes: D61; D71; H41.

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1 Introduction

We study a mechanism design problem in a binary excludable public good model. Each agent has a quasi-linear preference. A (direct) mechanism determines the set of agents who consume the public good and the cost share depending on agents' preferences.

We focus on mechanisms satisfying *strategy-proofness*, which requires that it be a dominant strategy for any agent to report his true preference. We evaluate strategy-proof mechanisms from the point of view of efficiency. However, it is well known that there exists no mechanism satisfying strategy-proofness and Pareto-efficiency [Holmström (1979)]. Hence, our aim is to design the most efficient mechanism possible.

Since the seminal work by Moulin (1994), several mechanisms are introduced and studied on this problem. There are two types of mechanisms: the exclusive type and the non-exclusive type. The exclusive type mechanism may exclude some agents from the consumption of the public good while the public good is supplied to the others. On the other hand, the non-exclusive type mechanism excludes no agent whenever the public good is supplied.

The Moulin mechanism¹ is a representative of the exclusive type ones. It has been known as the unique mechanism satisfying strategy-proofness, individual rationality, budget-balance, anonymity, and other desirable properties.² Furthermore, it minimizes the *supremal welfare loss* among the set of all mechanisms satisfying strategy-proofness, individual rationality, and the auxiliary axiom [Massó et al. (2015)]. The supremal welfare loss is the supremum of the welfare loss that occurs as a result of applying the mechanism, and is used as a measure of inefficiency.³

Ohseto (2005) has defined the non-exclusive type mechanism. The Ohseto mechanism⁴ achieves the smallest supremal welfare loss among the mechanisms satisfying strategy-proofness, budget-balance, and other desirable properties.⁵ Since the Moulin mechanism also satisfies these properties, this result means that the Ohseto mechanism is more efficient than the Moulin mechanism in terms of the supremal welfare loss.

This fact has an important implication as follows: it is a rational choice to exclude no agent from the consumption of the public good while we are able to exclude agents. Actually, in many countries, society does not exclude users of parks, roads, TV, and so on. Hence, we can theoretically support

¹This is also called as the equal cost sharing with maximal participation mechanism.

²See Moulin (1999), Deb and Razzolini (1999a, b), and Ohseto (2000).

³For example, see Juarez (2008), Moulin and Shenker (2001), and Ohseto (2010).

⁴This is also called as the optimal augmented serial mechanism.

⁵See Ohseto (2005) and Hashimoto and Saitoh (2016).

these policies.

Recently, the probabilistic mechanisms have been studied⁶ on this problem. Hashimoto and Shiozawa (2018) have defined probabilistic mechanism, called the α -mechanisms, and shown that the optimal α -mechanism achieves the smaller supremal welfare loss than that of the Ohseto mechanism. Since the optimal α -mechanism is a probabilistic and exclusive type mechanism, we have to study a probabilistic and non-exclusive type mechanism as well.

We define a new mechanism as a probabilistic and non-exclusive type one, called the *non-exclusive mechanism*. We first show that the non-exclusive mechanism satisfies strategy-proofness. Next, we establish that the non-exclusive mechanism is second-best efficient. Second best efficiency requires⁷ that the mechanism be on the Pareto-frontier of the set of strategy-proof mechanisms. Finally, we calculate the supremal welfare loss of the non-exclusive mechanism, and compare it with that of the several mechanisms.

The remainder of the paper is organized as follows. In Section 2, we set up the model. In Section 3, we define the basic properties. In Section 4, we introduce the mechanisms. In Section 5, we state our results. In Section 6, we provide the proofs.

2 Model

Let $N = \{1, 2, \dots, n\}$ be the set of agents. We consider the provision of a binary excludable public good $y \in \{0, 1\}$. The cost function $c(\cdot)$ of the excludable public good is normalized as follows: $c(0) = 0$ and $c(1) = 1$.

Each agent $i \in N$ has a preference for bundles consisting of a consumption level of the excludable public good $s_i \in \{0, 1\}$ and a cost share $t_i \in \mathbb{R}$. We assume that this preference is represented by a quasi-linear utility function, i.e., if agent i 's valuation of the excludable public good is $v_i \in \mathbb{R}_+$, then his utility for $(s_i, t_i) \in \{0, 1\} \times \mathbb{R}$ is

$$u_i((s_i, t_i); v_i) = s_i v_i - t_i.$$

A list $v \equiv (v_i)_{i \in N} \in \mathbb{R}_+^n$ is a valuation profile. Given $v \in \mathbb{R}_+^n$ and $N' \subset N$, $v_{N'} \in \mathbb{R}_+^{\#N'}$ and $v_{-N'} \in \mathbb{R}_+^{\#N \setminus N'}$ denote $(v_j)_{j \in N'}$ and $(v_j)_{j \notin N'}$, respectively.

The set of feasible allocations is

$$Z \equiv \left\{ (s_i, t_i)_{i \in N} \in (\{0, 1\} \times \mathbb{R})^n : \sum_{i \in N} t_i \geq \max_{i \in N} s_i \right\}.$$

⁶For example, see Dobzinski et al. (2018).

⁷In other words, it requires that the mechanism be undominated by the other strategy-proof mechanisms.

A deterministic mechanism is a function $f : \mathbb{R}_+^n \rightarrow Z$. Given a deterministic mechanism f and a valuation profile $v \in \mathbb{R}_+^n$, we denote agent i 's assignment under $f(v)$ as $f_i(v) \equiv (s_i(v), t_i(v)) \in \{0, 1\} \times \mathbb{R}$.

Let ΔZ be the set of all probability distributions on Z . Given a probability distribution, we denote by $\sigma_i \in [0, 1]$ and $\tau_i \in \mathbb{R}$ the probability that agent i consumes the excludable public good and the expected value that agent i pays under the probabilistic distribution, respectively. Each agent i evaluates (σ_i, τ_i) using his expected utility, i.e., if agent i 's valuation of the excludable public good is $v_i \in \mathbb{R}_+$, then his expected utility for $(\sigma_i, \tau_i) \in [0, 1] \times \mathbb{R}$ is

$$u_i((\sigma_i, \tau_i); v_i) = \sigma_i v_i - \tau_i.$$

A probabilistic mechanism is a function $\varphi : \mathbb{R}_+^n \rightarrow \Delta Z$. Given a probabilistic mechanism φ and a valuation profile $v \in \mathbb{R}_+^n$, we denote by $\sigma_i(v) \in [0, 1]$ and $\tau_i(v) \in \mathbb{R}$ the probability that agent i consumes the excludable public good and the expected value that agent i pays under the probability distribution $\varphi(v)$, respectively.

3 Properties

We define the basic properties. Since the deterministic mechanisms are special cases of the probabilistic mechanisms, we define the properties for the probabilistic mechanisms only.

Strategy-proofness states that it is a dominant strategy for any agent to report his true valuation.

Definition 1. A probabilistic mechanism φ satisfies **strategy-proofness** if for any $i \in N$, any $v \in \mathbb{R}_+^n$, and any $v'_i \in \mathbb{R}_+$,

$$\sigma_i(v)v_i - \tau_i(v) \geq \sigma_i(v'_i, v_{-i})v_i - \tau_i(v'_i, v_{-i}).$$

Second-best efficiency in individual (social) welfare states that the mechanism is on the Pareto-frontier in individual (social) welfare of the set of strategy-proof mechanisms.

Definition 2. A probabilistic mechanism φ^* is **second-best efficient in individual welfare** if there exists no strategy-proof probabilistic mechanism φ such that for any $v \in \mathbb{R}_+^n$ and any $i \in N$,

$$\sigma_i(v)v_i - \tau_i(v) \geq \sigma_i^*(v)v_i - \tau_i^*(v),$$

and for some $\hat{v} \in \mathbb{R}_+^n$ and some $j \in N$,

$$\sigma_j(\hat{v})\hat{v}_j - \tau_j(\hat{v}) > \sigma_j^*(\hat{v})\hat{v}_j - \tau_j^*(\hat{v}).$$

Definition 3. A probabilistic mechanism φ^* is **second-best efficient in social welfare** if there exists no strategy-proof probabilistic mechanism φ such that for any $v \in \mathbb{R}_+^n$,

$$\sum_{i \in N} (\sigma_i(v)v_i - \tau_i(v)) \geq \sum_{i \in N} (\sigma_i^*(v)v_i - \tau_i^*(v)),$$

and for some $\hat{v} \in \mathbb{R}_+^n$,

$$\sum_{i \in N} (\sigma_i(\hat{v})\hat{v}_i - \tau_i(\hat{v})) > \sum_{i \in N} (\sigma_i^*(\hat{v})\hat{v}_i - \tau_i^*(\hat{v})).$$

Remark 1. If a mechanism is second-best efficient in social welfare, then it is second-best efficient in individual welfare.

The *welfare loss* of a mechanism is the difference between the welfare of the first-best mechanism and that of the mechanism we consider.

Definition 4. Given a probabilistic mechanism φ and a valuation $v \in \mathbb{R}_+^n$, the **welfare loss** of φ at v is defined as follows:

$$WL(v; \varphi) \equiv \max \left\{ \sum_{k \in N} v_k - 1, 0 \right\} - \left(\sum_{k \in N} (\sigma_k(v)v_k - \tau_k(v)) \right).$$

The *supremal welfare loss* is the supremum of the welfare loss over all $v \in \mathbb{R}_+^n$.

Definition 5. Given a probabilistic mechanism φ , the **supremal welfare loss** of φ is

$$\sup_{v \in \mathbb{R}_+^n} WL(v; \varphi).$$

4 Mechanisms

We define several mechanisms. The non-exclusive mechanism is our new mechanism.

4.1 Ohseto mechanisms

First, we define⁸ the Ohseto mechanism. It works in a very simple way as follows: When there exists an agent whose valuation is greater than $\frac{1}{n}$, the public good is supplied to all agents and all agents share the cost $\frac{1}{n}$. Otherwise, the public good is not supplied and all agents pay nothing.

⁸Ohseto (2005) has defined the mechanisms that have the parameter $w \in N$. We only consider the mechanism with $w = 1$. Moreover, for simplicity of consideration, we adopt the definition that is welfare equivalent to original one.

Definition 6. A deterministic mechanism f^O is the Ohseto mechanism if for any $v \in \mathbb{R}_+^n$,

1. when $\{j \in N | v_j > \frac{1}{n}\} \neq \emptyset$, for any $i \in N$,

$$f_i^O(v) = (1, \frac{1}{n}),$$

2. when $\{j \in N | v_j > \frac{1}{n}\} = \emptyset$, for any $i \in N$,

$$f_i^O(v) = (0, 0).$$

4.2 Non-exclusive mechanism

We generalize the Ohseto mechanism to the probabilistic one. To do so, we consider asymmetric versions of the Ohseto mechanism. Define $C = \{c \in \mathbb{R}_+^n : \sum_{i \in N} c_i = 1\}$ as the set of cost share profiles.

Definition 7. Given $c \in C$. A deterministic mechanism f^c is defined as follows: for any $v \in \mathbb{R}_+^n$,

1. when $\{j \in N | v_j > c_j\} \neq \emptyset$, for any $i \in N$,

$$f_i^c(v) = (1, c_i),$$

2. when $\{j \in N | v_j > c_j\} = \emptyset$, for any $i \in N$,

$$f_i^c(v) = (0, 0).$$

Definition 8. A probabilistic mechanism φ^N is the **non-exclusive mechanism** if for any $v \in \mathbb{R}_+^n$, $\varphi^N(v)$ selects $f^c(v)$, where c follows the uniform distribution on C .

Remark 2. Notice that, when $\sum_{i \in N} v_i > 1$, for some agent $j \in N$, we must have $v_j > c_j$. Notice also that, when $\sum_{i \in N} v_i \leq 1$, the probability of occurrence of $c \in C$ such that for any $j \in N$, $v_i \leq c_j$, is $(1 - \sum_{k \in N} v_k)^{n-1}$. Thus, the probabilistic assignment of the non-exclusive mechanism can be summarized as follows: for any $v \in \mathbb{R}_+^n$ and any $i \in N$,

$$\sigma_i^N(v) = \begin{cases} 1 - (1 - \sum_{k \in N} v_k)^{n-1} & \text{if } \sum_{k \in N} v_k \leq 1, \\ 1 & \text{if } \sum_{k \in N} v_k > 1. \end{cases}$$

5 Results

We state our results. The proofs are in the next section. First, we state strategy-proofness of the new mechanism.

Theorem 1. The non-exclusive mechanism φ^N satisfies strategy-proofness.

Next, we give the results with regard to second-best efficiency of the new mechanism.

Theorem 2. The non-exclusive mechanism φ^N is second-best efficient in social welfare.

Remember that second-best efficiency in social welfare implies second-best efficiency in individual welfare. Thus, we immediately have the following.

Corollary 1. The non-exclusive mechanism φ^N is second-best efficient in individual welfare.

Finally, we calculate the supremal welfare loss of the new mechanism, and compare it with that of the mechanisms on the previous research.

Theorem 3. The supremal welfare loss of the non-exclusive mechanism φ^N is

$$\sup_{v \in \mathbb{R}_+^n} WL(v; \varphi^N) = \frac{n-1}{n} \left(\frac{1}{n} \right)^{\frac{1}{n-1}}.$$

Remark 3. For the Moulin mechanism f^M , the Ohseto mechanism f^O , and the optimal⁹ α -mechanism φ^α , the supremal welfare losses of these mechanisms have been known as follows:

$$\begin{aligned} \sup_{v \in \mathbb{R}_+^n} WL(v; f^M) &= \sum_{k=1}^n \frac{1}{k} - 1 \\ \sup_{v \in \mathbb{R}_+^n} WL(v; f^O) &= \frac{n-1}{n} \\ \sup_{v \in \mathbb{R}_+^n} WL(v; \varphi^\alpha) &= \max_{k \geq 1} \left\{ \sum_{h=1}^k \alpha_h \left(1 - \frac{k}{n} \right) \right\}. \end{aligned}$$

⁹The optimal α is calculated as follows: for any $k = 1, \dots, n$,

$$\alpha_k = 1 - \sum_{h=1}^{k-1} \alpha_h - \frac{1 - \frac{k}{n}}{\sum_{h=1}^k \frac{1}{h} + 1 - \frac{2k}{n}}. \quad (1)$$

Figure 1 exhibits these values for $n = 2, \dots, 100$. For any $n \geq 3$, we can obtain that

$$\sup_{v \in \mathbb{R}_+^n} WL(v; f^M) > \sup_{v \in \mathbb{R}_+^n} WL(v; f^O) > \sup_{v \in \mathbb{R}_+^n} WL(v; \varphi^N) > \sup_{v \in \mathbb{R}_+^n} WL(v; \varphi^\alpha).$$

Hence, we conclude that the non-exclusive mechanism φ^N is not superior to the optimal α -mechanism φ^α in the view point of the supremal welfare loss.

6 Proofs

6.1 Proof of Theorem 1

Let $i \in N$. Let $v \in \mathbb{R}_+^n$ and $v'_i \in \mathbb{R}_+$. For any $c \in C$, it is trivial that f^c is strategy-proof. Thus, it holds that

$$s_i^c(v)v_i - t_i^c(v) \geq s_i^c(v'_i, v_{-i})v_i - t_i^c(v'_i, v_{-i}).$$

Note that

$$\sigma_i^N(v) = E[s_i^c(v)]$$

and

$$\tau_i^N(v) = E[t_i^c(v)].$$

Thus, we have

$$\sigma_i^N(v)v_i - \tau_i^N(v) \geq \sigma_i^N(v'_i, v_{-i})v_i - \tau_i^N(v'_i, v_{-i}).$$

Therefore, Theorem 1 is valid.

6.2 Myerson's Lemma

To prove Theorem 2, we use the following Lemma, proved by Myerson (1981) in a similar model.

Lemma (Myerson 1981). If a mechanism φ satisfies strategy-proofness, then for any $i \in N$, any $v_i, v'_i \in \mathbb{R}_+$ such that $v_i \leq v'_i$, and any $v_{-i} \in \mathbb{R}_+^{n-1}$, it holds that

$$\sigma_i(v_i, v_{-i}) \leq \sigma_i(v'_i, v_{-i})$$

and

$$\sigma_i(v'_i, v_{-i})v'_i - \tau_i(v'_i, v_{-i}) = \sigma_i(v_i, v_{-i})v_i - \tau_i(v_i, v_{-i}) + \int_{v_i}^{v'_i} \sigma_i(x_i, v_{-i})dx_i.$$

6.3 Proof of Theorem 2

Let φ be a strategy-proof probabilistic mechanism such that for any $v \in \mathbb{R}_+^n$,

$$\sum_{i \in N} (\sigma_i(v) v_i - \tau_i(v)) \geq \sum_{i \in N} (\sigma_i^N(v) v_i - \tau_i^N(v)).$$

We show that for any $v \in \mathbb{R}_+^n$,

$$\sum_{i \in N} (\sigma_i(v) v_i - \tau_i(v)) = \sum_{i \in N} (\sigma_i^N(v) v_i - \tau_i^N(v)).$$

For any $S \subset N$, define $V(S) = \{v \in \mathbb{R}_+^n \mid \text{if } i \notin S, v_i = 0\}$. We consider the following mathematical induction.

Base case: For any $S \subset N$ such that $\#S = 0$, and any $v \in V(S)$, it holds that

$$\sum_{i \in N} (\sigma_i(v) v_i - \tau_i(v)) = \sum_{i \in N} (\sigma_i^N(v) v_i - \tau_i^N(v)).$$

Proof of Base case. Let $S \subset N$ be such that $\#S = 0$. Let $v \in V(S)$, that is, $v = (0, \dots, 0)$. Since, when $v = (0, \dots, 0)$, $\varphi^N(v)$ is Pareto-efficient, we have the desired result. $\square$

Induction step: Suppose that for any $S \subset N$ such that $\#S = t - 1$, and any $v \in V(S)$, it holds that

$$\sum_{i \in N} (\sigma_i(v) v_i - \tau_i(v)) = \sum_{i \in N} (\sigma_i^N(v) v_i - \tau_i^N(v)).$$

Then, for any $S \subset N$ such that $\#S = t$, and any $v \in V(S)$, it holds that

$$\sum_{i \in N} (\sigma_i(v) v_i - \tau_i(v)) = \sum_{i \in N} (\sigma_i^N(v) v_i - \tau_i^N(v)).$$

Proof of Induction step. Let $S \subset N$ be such that $\#S = t$. Suppose to the contrary that for some $\hat{v} \in V(S)$,

$$\sum_{i \in N} (\sigma_i(\hat{v}) \hat{v}_i - \tau_i(\hat{v})) > \sum_{i \in N} (\sigma_i^N(\hat{v}) \hat{v}_i - \tau_i^N(\hat{v})).$$

Claim 1. For any $v \in \mathbb{R}_+^n$ such that $\sum_{i \in N} v_i \neq 1$, and any $i \in N$, it holds that

$$\sigma_i(v) \leq \sigma_i^N(v).$$

Proof of Claim 1. Suppose to the contrary that for some $v \in \mathbb{R}_+^n$ such that $\sum_{i \in N} v_i \neq 1$, and some $j \in N$, it holds that

$$\sigma_j(v) > \sigma_j^N(v).$$

Then, it holds that

$$\max_{i \in N} \sigma_i(v) \geq \sigma_j(v) > \sigma_j^N(v) = \max_{i \in N} \sigma_i^N(v).$$

When $\sum_{i \in N} v_i > 1$, we must have $\sigma_j^N(v) = 1$, which means a contradiction. Hence, it follows that $\sum_{i \in N} v_i < 1$. Since, by feasibility, $\sum_{i \in N} \tau_i(v) \geq \max_{i \in N} \sigma_i(v)$ and $\sum_{i \in N} \tau_i^N(v) = \max_{i \in N} \sigma_i^N(v)$, it follows that

$$\begin{aligned} \sum_{i \in N} (\sigma_i(v)v_i - \tau_i(v)) &\leq \max_{i \in N} \sigma_i(v) \sum_{i \in N} v_i - \max_{i \in N} \sigma_i(v) \\ &= \max_{i \in N} \sigma_i(v) \left(\sum_{i \in N} v_i - 1 \right) \\ &< \max_{i \in N} \sigma_i^N(v) \left(\sum_{i \in N} v_i - 1 \right) \\ &= \sum_{i \in N} (\sigma_i^N(v)v_i - \tau_i^N(v)), \end{aligned}$$

which is a contradiction. $\square$

Claim 2. $\sum_{i \in N} \hat{v}_i < 1$.

Proof of Claim 2. Since, when $\sum_{i \in N} v_i \geq 1$, $\varphi^N(v)$ is Pareto-efficient, we have the desired result. $\square$

Claim 3. For some $j \in N$, $\sigma_j(\hat{v}) < \sigma_j^N(\hat{v})$.

Proof of Claim 3. Suppose to the contrary that for any $i \in N$, $\sigma_i(\hat{v}) \geq \sigma_i^N(\hat{v})$. Then, it holds that

$$\max_{i \in N} \sigma_i(\hat{v}) \geq \max_{i \in N} \sigma_i^N(\hat{v}).$$

Since $\sum_{i \in N} \tau_i(\hat{v}) \geq \max_{i \in N} \sigma_i(\hat{v})$ and $\sum_{i \in N} \tau_i^N(\hat{v}) = \max_{i \in N} \sigma_i^N(\hat{v})$, by Claim 2, it follows that

$$\begin{aligned} \sum_{i \in N} (\sigma_i(\hat{v})\hat{v}_i - \tau_i(\hat{v})) &\leq \max_{i \in N} \sigma_i(\hat{v}) \sum_{i \in N} \hat{v}_i - \max_{i \in N} \sigma_i(\hat{v}) \\ &= \max_{i \in N} \sigma_i(\hat{v}) \left(\sum_{i \in N} \hat{v}_i - 1 \right) \\ &\leq \max_{i \in N} \sigma_i^N(\hat{v}) \left(\sum_{i \in N} \hat{v}_i - 1 \right) \\ &= \sum_{i \in N} (\sigma_i^N(\hat{v})\hat{v}_i - \tau_i^N(\hat{v})), \end{aligned}$$

which is a contradiction. $\square$

For simplicity of notation, we assume that $\sigma_1(\hat{v}) < \sigma_1^N(\hat{v})$.

Claim 4. It holds that¹⁰

$$\begin{aligned} & \sigma_1(1, \hat{v}_{-\{1\}}) \cdot 1 - \tau_1(1, \hat{v}_{-\{1\}}) + \sum_{i \neq 1} \left(\sigma_i(0, 1, \hat{v}_{-\{1,i\}}) \cdot 1 - \tau_i(0, 1, \hat{v}_{-\{1,i\}}) \right) \\ & < \sigma_1^N(1, \hat{v}_{-\{1\}}) \cdot 1 - \tau_1^N(1, \hat{v}_{-\{1\}}) + \sum_{i \neq 1} \left(\sigma_i^N(0, 1, \hat{v}_{-\{1,i\}}) \cdot 1 - \tau_i^N(0, 1, \hat{v}_{-\{1,i\}}) \right). \end{aligned}$$

Proof of Claim 4. By Myerson's Lemma and Claims 1 and 3, we have¹¹

$$\begin{aligned} & \sigma_1(1, \hat{v}_{-\{1\}}) \cdot 1 - \tau_1(1, \hat{v}_{-\{1\}}) \\ & = \sigma_1(0, \hat{v}_{-\{1\}}) \cdot 0 - \tau_1(0, \hat{v}_{-\{1\}}) + \int_0^1 \sigma_1(x_1, \hat{v}_{-\{1\}}) dx_1 \\ & < \sigma_1(0, \hat{v}_{-\{1\}}) \cdot 0 - \tau_1(0, \hat{v}_{-\{1\}}) + \int_0^1 \sigma_1^N(x_1, \hat{v}_{-\{1\}}) dx_1 \end{aligned}$$

and for any $i \neq 1$,

$$\begin{aligned} & \sigma_i(0, 1, \hat{v}_{-\{1,i\}}) \cdot 1 - \tau_i(0, 1, \hat{v}_{-\{1,i\}}) \\ & = \sigma_i(0, \hat{v}_i, \hat{v}_{-\{1,i\}}) \cdot \hat{v}_i - \tau_i(0, \hat{v}_i, \hat{v}_{-\{1,i\}}) + \int_{\hat{v}_i}^1 \sigma_i(0, x_i, \hat{v}_{-\{1,i\}}) dx_i \\ & \leq \sigma_i(0, \hat{v}_i, \hat{v}_{-\{1,i\}}) \cdot \hat{v}_i - \tau_i(0, \hat{v}_i, \hat{v}_{-\{1,i\}}) + \int_{\hat{v}_i}^1 \sigma_i^N(0, x_i, \hat{v}_{-\{1,i\}}) dx_i. \end{aligned}$$

By summing up these inequalities, it holds that

$$\begin{aligned} & \sigma_1(1, \hat{v}_{-\{1\}}) \cdot 1 - \tau_1(1, \hat{v}_{-\{1\}}) + \sum_{i \neq 1} \left(\sigma_i(0, 1, \hat{v}_{-\{1,i\}}) \cdot 1 - \tau_i(0, 1, \hat{v}_{-\{1,i\}}) \right) \\ & < \sigma_1(0, \hat{v}_{-\{1\}}) \cdot 0 - \tau_1(0, \hat{v}_{-\{1\}}) + \sum_{i \neq 1} \left(\sigma_i(0, \hat{v}_i, \hat{v}_{-\{1,i\}}) \cdot \hat{v}_i - \tau_i(0, \hat{v}_i, \hat{v}_{-\{1,i\}}) \right) \\ & \quad + \int_0^1 \sigma_1^N(x_1, \hat{v}_{-\{1\}}) dx_1 + \sum_{i \neq 1} \int_{\hat{v}_i}^1 \sigma_i^N(0, x_i, \hat{v}_{-\{1,i\}}) dx_i. \end{aligned}$$

¹⁰In the profile $(0, 1, \hat{v}_{-\{1,i\}})$, the first element means agent 1's valuation, and the second element means agent i 's valuation.

¹¹Since $\sigma_1^N(v)$ is continuous and $\sigma_1(v)$ is non-decreasing, $\sigma_1(\hat{v}) < \sigma_1^N(\hat{v})$ implies that there exists a sufficiently small $\epsilon > 0$ such that for any $x_1 \in [\hat{v}_1 - \epsilon, \hat{v}_1]$, $\sigma_1(x_1, \hat{v}_{-\{1\}}) < \sigma_1^N(x_1, \hat{v}_{-\{1\}})$.

Then, by the induction hypothesis, it follows that

$$\begin{aligned} & \sigma_1(0, \hat{v}_{-\{1\}}) \cdot 0 - \tau_1(0, \hat{v}_{-\{1\}}) + \sum_{i \neq 1} \left(\sigma_i(0, \hat{v}_i, \hat{v}_{-\{1,i\}}) \cdot \hat{v}_i - \tau_i(0, \hat{v}_i, \hat{v}_{-\{1,i\}}) \right) \\ &= \sigma_1^N(0, \hat{v}_{-\{1\}}) \cdot 0 - \tau_1^N(0, \hat{v}_{-\{1\}}) + \sum_{i \neq 1} \left(\sigma_i^N(0, \hat{v}_i, \hat{v}_{-\{1,i\}}) \cdot \hat{v}_i - \tau_i^N(0, \hat{v}_i, \hat{v}_{-\{1,i\}}) \right). \end{aligned}$$

Thus, by Myerson's Lemma, we have

$$\begin{aligned} & \sigma_1(1, \hat{v}_{-\{1\}}) \cdot 1 - \tau_1(1, \hat{v}_{-\{1\}}) + \sum_{i \neq 1} \left(\sigma_i(0, 1, \hat{v}_{-\{1,i\}}) \cdot 1 - \tau_i(0, 1, \hat{v}_{-\{1,i\}}) \right) \\ & < \sigma_1^N(0, \hat{v}_{-\{1\}}) \cdot 0 - \tau_1^N(0, \hat{v}_{-\{1\}}) + \sum_{i \neq 1} \left(\sigma_i^N(0, \hat{v}_i, \hat{v}_{-\{1,i\}}) \cdot \hat{v}_i - \tau_i^N(0, \hat{v}_i, \hat{v}_{-\{1,i\}}) \right) \\ & \quad + \int_0^1 \sigma_1^N(x_1, \hat{v}_{-\{1\}}) dx_1 + \sum_{i \neq 1} \int_{\hat{v}_i}^1 \sigma_i^N(0, x_i, \hat{v}_{-\{1,i\}}) dx_i \\ &= \sigma_1^N(1, \hat{v}_{-\{1\}}) \cdot 1 - \tau_1^N(1, \hat{v}_{-\{1\}}) + \sum_{i \neq 1} \left(\sigma_i^N(0, 1, \hat{v}_{-\{1,i\}}) \cdot 1 - \tau_i^N(0, 1, \hat{v}_{-\{1,i\}}) \right). \end{aligned}$$

□

Claim 5. We have the following equation:

$$\begin{aligned} & \sigma_1(1, \hat{v}_{-\{1\}}) \cdot 1 - \tau_1(1, \hat{v}_{-\{1\}}) + \sum_{i \neq 1} \left(\sigma_i(0, 1, \hat{v}_{-\{1,i\}}) \cdot 1 - \tau_i(0, 1, \hat{v}_{-\{1,i\}}) \right) \\ &= n - 1. \end{aligned}$$

Proof of Claim 5. See Appendix. □

Claim 6. We have the following equation:

$$\begin{aligned} & \sigma_1^N(1, \hat{v}_{-\{1\}}) \cdot 1 - \tau_1^N(1, \hat{v}_{-\{1\}}) + \sum_{i \neq 1} \left(\sigma_i^N(0, 1, \hat{v}_{-\{1,i\}}) \cdot 1 - \tau_i^N(0, 1, \hat{v}_{-\{1,i\}}) \right) \\ &= n - 1. \end{aligned}$$

Proof of Claim 6. See Appendix. □

By Claims 4, 5, and 6, we have a contradiction. Thus, Induction step is valid. □

Therefore, the non-exclusive mechanism φ^N is second-best efficient in social welfare. □

6.4 Proof of Theorem 3

When $\sum_{k \in N} v_k \geq 1$, since $\varphi^N(v)$ is Pareto-efficient, the welfare loss is zero. Hence, we consider the case $\sum_{k \in N} v_k < 1$. Remember that for any $i \in N$,

$$\sigma_i^N(v) = 1 - \left(1 - \sum_{k \in N} v_k\right)^{n-1}.$$

Then, we have

$$\begin{aligned} WL(v; \varphi^N) &= 0 - \left(\sum_{k \in N} \left(\sigma_k^N(v) v_k - \tau_k^N(v) \right) \right) \\ &= -\sigma_i^N(v) \left(\sum_{k \in N} v_k - 1 \right) \\ &= \left(1 - \left(1 - \sum_{k \in N} v_k \right)^{n-1} \right) \left(1 - \sum_{k \in N} v_k \right). \end{aligned}$$

Set $X = 1 - \sum_{k \in N} v_k$ and rewrite as

$$\begin{aligned} WL(v; \varphi^N) &= (1 - X^{n-1})X \\ &= X - X^n. \end{aligned}$$

Then, the first-order condition is

$$\begin{aligned} \frac{dWL(v; \varphi^N)}{dX} &= 1 - nX^{n-1} \\ &= 0. \end{aligned}$$

Hence, the welfare loss reaches the maximum at

$$X^* = \left(\frac{1}{n} \right)^{\frac{1}{n-1}}$$

and the maximal value is

$$\frac{n-1}{n} \left(\frac{1}{n} \right)^{\frac{1}{n-1}}.$$

□

Appendix

We prove Claim 5 in the proof of Theorem 2. The same argument holds for Claim 6 in the proof of Theorem 2. Notice that

$$\sigma_1(1, \hat{v}_{-\{1\}}) - \tau_1(1, \hat{v}_{-\{1\}}) = \sum_{i \neq 1} \tau_i(1, \hat{v}_{-\{1\}})$$

and for any $i \neq 1$,

$$\sigma_i(0, 1, \hat{v}_{-\{1,i\}}) - \tau_i(0, 1, \hat{v}_{-\{1,i\}}) = \sum_{k \neq i} \tau_k(0, 1, \hat{v}_{-\{1,i\}}).$$

Then, by strategy-proofness, for any $i \neq 1$, we have¹²

$$\tau_i(1, \hat{v}_{-\{1\}}) = \tau_i(1, 1, \hat{v}_{-\{1,i\}}),$$

and for any $i \neq 1$ and any $k \neq 1, i$, we have

$$\tau_k(0, 1, \hat{v}_{-\{1,i\}}) = \tau_k(0, 1, 1, \hat{v}_{-\{1,i,k\}})$$

and

$$\tau_1(0, 1, \hat{v}_{-\{1,i\}}) = \tau_1(1, 1, \hat{v}_{-\{1,i\}}).$$

Hence, it holds that

$$\begin{aligned} & \sigma_1(1, \hat{v}_{-\{1\}}) \cdot 1 - \tau_1(1, \hat{v}_{-\{1\}}) + \sum_{i \neq 1} \left(\sigma_i(0, 1, \hat{v}_{-\{1,i\}}) \cdot 1 - \tau_i(0, 1, \hat{v}_{-\{1,i\}}) \right) \\ &= \sum_{i \neq 1} \tau_i(1, \hat{v}_{-\{1\}}) + \sum_{i \neq 1} \left(\sum_{k \neq 1, i} \tau_k(0, 1, \hat{v}_{-\{1,i\}}) + \tau_1(0, 1, \hat{v}_{-\{1,i\}}) \right) \\ &= \sum_{i \neq 1} \tau_i(1, 1, \hat{v}_{-\{1,i\}}) + \sum_{i \neq 1} \left(\sum_{k \neq 1, i} \tau_k(0, 1, 1, \hat{v}_{-\{1,i,k\}}) + \tau_1(1, 1, \hat{v}_{-\{1,i\}}) \right) \\ &= \sum_{i \neq 1} \left(\tau_1(1, 1, \hat{v}_{-\{1,i\}}) + \tau_i(1, 1, \hat{v}_{-\{1,i\}}) \right) + \sum_{i \neq 1} \sum_{k \neq 1, i} \tau_k(0, 1, 1, \hat{v}_{-\{1,i,k\}}). \end{aligned}$$

Notice that for any $i \neq 1$,

$$\tau_1(1, 1, \hat{v}_{-\{1,i\}}) + \tau_i(1, 1, \hat{v}_{-\{1,i\}}) = 1 - \sum_{k \neq 1, i} \tau_k(1, 1, \hat{v}_{-\{1,i\}})$$

¹²By Myerson's Lemma, we have

$$\begin{aligned} & \sigma_i(1, 1, \hat{v}_{-\{1,i\}}) \cdot 1 - \tau_i(1, 1, \hat{v}_{-\{1,i\}}) \\ &= \sigma_i(1, \hat{v}_i, \hat{v}_{-\{1,i\}}) \cdot \hat{v}_i - \tau_i(1, \hat{v}_i, \hat{v}_{-\{1,i\}}) + \int_{\hat{v}_i}^1 \sigma_i(1, x_i, \hat{v}_{-\{1,i\}}) dx_i. \end{aligned}$$

Note that for any $x_i > 0$, we have $\sigma_i(1, x_i, \hat{v}_{-\{1,i\}}) = 1$. This means that

$$\sigma_i(1, 1, \hat{v}_{-\{1,i\}}) \cdot 1 = \sigma_i(1, \hat{v}_i, \hat{v}_{-\{1,i\}}) \cdot \hat{v}_i + \int_{\hat{v}_i}^1 \sigma_i(1, x_i, \hat{v}_{-\{1,i\}}) dx_i.$$

Hence, we have $\tau_i(1, 1, \hat{v}_{-\{1,i\}}) = \tau_i(1, \hat{v}_i, \hat{v}_{-\{1,i\}})$.

and for any $i \neq 1$ and any $k \neq 1, i$,

$$\tau_i(0, 1, 1, \hat{v}_{-\{1,i,k\}}) + \tau_k(0, 1, 1, \hat{v}_{-\{1,i,k\}}) = 1 - \sum_{h \neq i,k} \tau_h(0, 1, 1, \hat{v}_{-\{1,i,k\}}).$$

Then, by strategy-proofness, for any $i \neq 1$ and any $h \neq 1, i$, we have

$$\tau_h(1, 1, \hat{v}_{-\{1,i\}}) = \tau_h(1, 1, 1, \hat{v}_{-\{1,i,h\}}),$$

and for any $i \neq 1$, any $k \neq 1, i$, and any $h \neq 1, i, k$, we have

$$\tau_h(0, 1, 1, \hat{v}_{-\{1,i,k\}}) = \tau_h(0, 1, 1, 1, \hat{v}_{-\{1,i,k,h\}})$$

and

$$\tau_1(0, 1, 1, \hat{v}_{-\{1,i,k\}}) = \tau_1(1, 1, 1, \hat{v}_{-\{1,i,k\}}).$$

Hence, it holds that

$$\begin{aligned} & \sigma_1(1, \hat{v}_{-\{1\}}) \cdot 1 - \tau_1(1, \hat{v}_{-\{1\}}) + \sum_{i \neq 1} \left(\sigma_i(0, 1, \hat{v}_{-\{1,i\}}) \cdot 1 - \tau_i(0, 1, \hat{v}_{-\{1,i\}}) \right) \\ &= \sum_{i \neq 1} \left(\tau_1(1, 1, \hat{v}_{-\{1,i\}}) + \tau_i(1, 1, \hat{v}_{-\{1,i\}}) \right) + \sum_{i \neq 1} \sum_{k \neq 1,i} \tau_k(0, 1, 1, \hat{v}_{-\{1,i,k\}}) \\ &= \sum_{i \neq 1} \left(1 - \sum_{k \neq 1,i} \tau_k(1, 1, \hat{v}_{-\{1,i\}}) \right) + \sum_{\substack{i \neq 1 \\ i < k}} \left(1 - \sum_{h \neq i,k} \tau_h(0, 1, 1, \hat{v}_{-\{1,i,k\}}) \right) \\ &= \sum_{i \neq 1} \left(1 - \sum_{k \neq 1,i} \tau_k(1, 1, 1, \hat{v}_{-\{1,i,k\}}) \right) \\ &\quad + \sum_{\substack{i \neq 1 \\ i < k}} \left(1 - \sum_{h \neq 1,i,k} \tau_h(0, 1, 1, 1, \hat{v}_{-\{1,i,k,h\}}) - \tau_1(1, 1, 1, \hat{v}_{-\{1,i,k\}}) \right) \\ &= \sum_{i \neq 1} \left(1 - \sum_{\substack{i \neq 1 \\ i < k}} \left(\tau_1(1, 1, 1, \hat{v}_{-\{1,i,k\}}) + \tau_i(1, 1, 1, \hat{v}_{-\{1,i,k\}}) + \tau_k(1, 1, 1, \hat{v}_{-\{1,i,k\}}) \right) \right) \\ &\quad + \sum_{\substack{i \neq 1 \\ i < k}} \left(1 - \sum_{h \neq 1,i,k} \tau_h(0, 1, 1, 1, \hat{v}_{-\{1,i,k,h\}}) \right). \end{aligned}$$

Notice that for any $i, k \neq 1$ such that $i \neq k$,

$$\tau_1(1, 1, 1, \hat{v}_{-\{1,i,k\}}) + \tau_i(1, 1, 1, \hat{v}_{-\{1,i,k\}}) + \tau_k(1, 1, 1, \hat{v}_{-\{1,i,k\}}) = 1 - \sum_{h \neq 1,i,k} \tau_h(1, 1, 1, \hat{v}_{-\{1,i,k\}})$$

and for any $i \neq 1$ and any $k \neq 1, i$,

$$\begin{aligned} & \tau_i(0, 1, 1, 1, \hat{v}_{-\{1,i,k,h\}}) + \tau_k(0, 1, 1, 1, \hat{v}_{-\{1,i,k,h\}}) + \tau_h(0, 1, 1, 1, \hat{v}_{-\{1,i,k,h\}}) \\ &= 1 - \sum_{\ell \neq i,k,h} \tau_\ell(0, 1, 1, 1, \hat{v}_{-\{1,i,k,h\}}). \end{aligned}$$

Then, by strategy-proofness, for any distinct $i, k, \ell \neq 1$, we have

$$\tau_\ell(1, 1, 1, \hat{v}_{-\{1,i,k\}}) = \tau_\ell(1, 1, 1, 1, \hat{v}_{-\{1,i,k,\ell\}}),$$

and for any distinct $i, k, h, \ell \neq 1$, we have

$$\tau_\ell(0, 1, 1, 1, \hat{v}_{-\{1,i,k,h\}}) = \tau_\ell(0, 1, 1, 1, 1, \hat{v}_{-\{1,i,k,h,\ell\}})$$

and

$$\tau_1(0, 1, 1, 1, \hat{v}_{-\{1,i,k,h\}}) = \tau_1(1, 1, 1, 1, \hat{v}_{-\{1,i,k,h\}}).$$

Hence, it holds that

$$\begin{aligned} & \sigma_1(1, \hat{v}_{-\{1\}}) \cdot 1 - \tau_1(1, \hat{v}_{-\{1\}}) + \sum_{i \neq 1} \left(\sigma_i(0, 1, \hat{v}_{-\{1,i\}}) \cdot 1 - \tau_i(0, 1, \hat{v}_{-\{1,i\}}) \right) \\ &= \sum_{i \neq 1} (1) - \sum_{\substack{i \neq 1 \\ i < k}} \left(\tau_1(1, 1, 1, \hat{v}_{-\{1,i,k\}}) + \tau_i(1, 1, 1, \hat{v}_{-\{1,i,k\}}) + \tau_k(1, 1, 1, \hat{v}_{-\{1,i,k\}}) \right) \\ & \quad + \sum_{\substack{i \neq 1 \\ i < k}} \left(1 - \sum_{h \neq 1, i, k} \tau_h(0, 1, 1, 1, \hat{v}_{-\{1,i,k,h\}}) \right) \\ &= \sum_{i \neq 1} (1) - \sum_{\substack{i \neq 1 \\ i < k}} \left(1 - \sum_{h \neq 1, i, k} \tau_h(1, 1, 1, \hat{v}_{-\{1,i,k\}}) \right) \\ & \quad + \sum_{\substack{i \neq 1 \\ i < k}} (1) - \sum_{\substack{i \neq 1 \\ i < k < h}} \left(1 - \sum_{\ell \neq i, k, h} \tau_\ell(0, 1, 1, 1, \hat{v}_{-\{1,i,k,h\}}) \right) \\ &= \sum_{i \neq 1} (1) - \sum_{\substack{i \neq 1 \\ i < k}} \left(1 - \sum_{h \neq 1, i, k} \tau_h(1, 1, 1, 1, \hat{v}_{-\{1,i,k,\ell\}}) \right) \\ & \quad + \sum_{\substack{i \neq 1 \\ i < k}} (1) - \sum_{\substack{i \neq 1 \\ i < k < h}} \left(1 - \sum_{\ell \neq 1, i, k, h} \tau_\ell(0, 1, 1, 1, 1, \hat{v}_{-\{1,i,k,h,\ell\}}) \right. \\ & \quad \left. - \tau_1(1, 1, 1, 1, \hat{v}_{-\{1,i,k,h\}}) \right) \Big) \\ &= \sum_{i \neq 1} (1) + \sum_{\substack{i \neq 1 \\ i < k < h}} \left(\tau_1(1, 1, 1, 1, \hat{v}_{-\{1,i,k,h\}}) + \tau_i(1, 1, 1, 1, \hat{v}_{-\{1,i,k,h\}}) \right. \\ & \quad \left. + \tau_k(1, 1, 1, 1, \hat{v}_{-\{1,i,k,h\}}) + \tau_h(1, 1, 1, 1, \hat{v}_{-\{1,i,k,\ell\}}) \right) \\ & \quad - \sum_{\substack{i \neq 1 \\ i < k < h}} \left(1 - \sum_{\ell \neq 1, i, k, h} \tau_\ell(0, 1, 1, 1, 1, \hat{v}_{-\{1,i,k,h,\ell\}}) \right). \end{aligned}$$

By repeating the similar argument, we have

$$\begin{aligned} & \sigma_1(1, \hat{v}_{-\{1\}}) \cdot 1 - \tau_1(1, \hat{v}_{-\{1\}}) + \sum_{i \neq 1} \left(\sigma_i(0, 1, \hat{v}_{-\{1, i\}}) \cdot 1 - \tau_i(0, 1, \hat{v}_{-\{1, i\}}) \right) \\ &= \sum_{i \neq 1} (1) + (-1)^{n-2} \left(\tau_1(1, \dots, 1) + \dots + \tau_n(1, \dots, 1) \right) - (-1)^{n-2} 1. \end{aligned}$$

Since $\tau_1(1, \dots, 1) + \dots + \tau_n(1, \dots, 1) = 1$, it holds that

$$\begin{aligned} & \sigma_1(1, \hat{v}_{-\{1\}}) \cdot 1 - \tau_1(1, \hat{v}_{-\{1\}}) + \sum_{i \neq 1} \left(\sigma_i(0, 1, \hat{v}_{-\{1, i\}}) \cdot 1 - \tau_i(0, 1, \hat{v}_{-\{1, i\}}) \right) \\ &= n - 1. \end{aligned}$$

Therefore, Claim 5 in the proof of Theorem 2 is valid. $\square$

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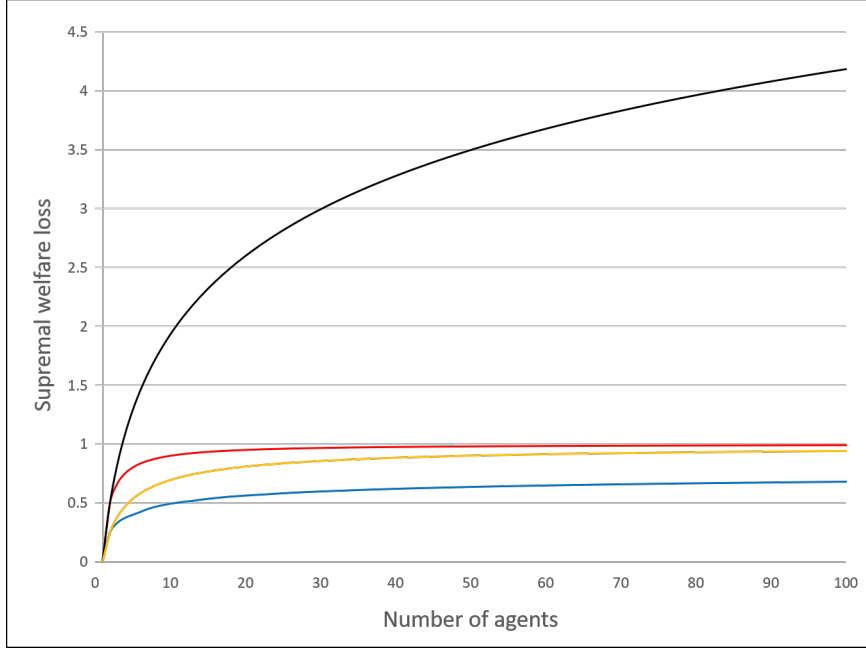


Figure 1: The supremal welfare losses of the four mechanisms: the Moulin mechanism f^M [Black], the Ohseto mechanism f^O [Red], and the non-exclusive mechanism φ^N [Yellow], the optimal α -mechanism φ^α [Blue].