

Strategy-Proofness and Pareto-Efficiency in the Income Redistribution Problem with Inequality Aversion

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Abstract

We consider the income redistribution problem among agents. Each agent has an inequity-averse preference represented by a utility function proposed by Fehr and Schmidt (1999). We focus on rules satisfying *strategy-proofness*, *Pareto-efficiency*, and *individual rationality*. We introduce two rules satisfying these properties, the *gross uniform redistribution rule* and the *net uniform redistribution rule*. We characterize these rules by *strategy-proofness*, *Pareto-efficiency*, *individual rationality*, and additional axioms.

1 Introduction

We consider the income redistribution problem among agents. Each agent has the initial income (endowment) and has a preference represented by a utility function proposed by Fehr and Schmidt (1999). The agent who has such a preference cares about not only his income assignment, but also other agents' ones. In other words, he is inequality-averse. A rule determines the income redistribution depending on agents' preferences.

We focus on rules satisfying *strategy-proofness*, *Pareto-efficiency*, and *individual rationality*. *Strategy-proofness* requires that it be a dominant strategy for any agent to report his true preference. *Pareto-efficiency* requires that the rule always assign a Pareto-efficient allocation. *Individual rationality* requires that the rule always assign an allocation at least as well as the initial income distribution (endowment).

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We introduce two rules satisfying *strategy-proofness*, *Pareto-efficiency*, and *individual rationality*. On these rules, the agents who are rich and strongly inequality-averse redistribute some part of their income to other agents. The *gross uniform redistribution rule* redistributes income uniformly to the poorest agents.¹ The *net uniform redistribution rule* redistributes income uniformly to all the other agents.²

We characterize these rules by *strategy-proofness*, *Pareto-efficiency*, *individual rationality*, and additional axioms. The *gross uniform redistribution rule* is the only rule satisfying *strategy-proofness*, *Pareto-efficiency*, *individual rationality*, and *envy-freeness for recipients*. *Envy-freeness for recipients* requires that no agent prefer any recipient's position to his one.³ The *net uniform redistribution rule* is the only rule satisfying *strategy-proofness*, *Pareto-efficiency*, *individual rationality*, and *envy-freeness on non-negative net trade*. *Envy-freeness on non-negative net trade* requires that no agent prefer any agent's position in the view of the non-negative net trades to his one.⁴

The remainder of the paper is organized as follows. In Section 2, we set up the model. In Section 3, we define the properties. In Section 4, we introduce the rules. In Section 5, we state our results. In Appendix, we provide the proofs.

2 Model

Let $N = \{1, 2, \dots, n\}$ be the set of agents. We consider the income redistribution problem among agents. Each agent $i \in N$ has the initial income (endowment) $e_i \in \mathbb{R}_+$. Without loss of generality, we assume that $e_1 \geq e_2 \geq \dots \geq e_n$, and normalize $\sum_{i \in N} e_i = 1$. The set of feasible allocations is

$$X \equiv \left\{ (x_1, \dots, x_n) \in \mathbb{R}_+^n : \sum_{i \in N} x_i = 1 \right\}.$$

Each agent $i \in N$ has a preference on X . We assume that this preference is represented by a utility function proposed by Fehr and Schmidt (1999),

¹A similar rule has defined in Hashimoto and Wakayama (2021) in reallocation problem with single-peaked preferences.

²A similar rule has defined in Klaus, Peters, and Storcken (1997) in reallocation problem with single-peaked preferences.

³This property is a weaker form of *envy-freeness*, which is first introduced by Foley (1967).

⁴The notion of *envy-freeness* in terms of changes in allocations is first formulated by Schmeidler and Vind (1972) in the context of pure exchange economies.

i.e.,

$$u_i(x; \alpha_i, \beta_i) = x_i - \alpha_i \frac{1}{n-1} \sum_{j \in N} \max\{x_j - x_i, 0\} - \beta_i \frac{1}{n-1} \sum_{j \in N} \max\{x_i - x_j, 0\},$$

where $\alpha_i \geq \beta_i$ and $\beta_i \in [0, 1)$. Let $\mathcal{U}$ be the set of the utility functions (or preferences) proposed by Fehr and Schmidt (1999).

A list $u \equiv (u_i)_{i \in N} \in \mathcal{U}^n$ is a preference profile. Given $u \in \mathcal{U}^n$ and $N' \subset N$, $u_{N'} \in \mathcal{U}^{\#N'}$ and $u_{-N'} \in \mathcal{U}^{\#N \setminus N'}$ denote $(u_j)_{j \in N'}$ and $(u_j)_{j \notin N'}$, respectively.

A rule is a function $f : \mathcal{U}^n \rightarrow X$. Given a rule f and a preference profile $u \in \mathcal{U}^n$, we denote agent i 's assignment under $f(u)$ as $f_i(u) \in \mathbb{R}_+$.

3 Properties

3.1 Basic properties

We define the basic properties. *Strategy-proofness* states that it is a dominant strategy for any agent to report his true valuation.

Definition 1. A rule f satisfies **strategy-proofness** if for any $i \in N$, any $u \in \mathcal{U}^n$, and any $u'_i \in \mathcal{U}$, it holds that

$$u_i(f(u)) \geq u_i(f(u'_i, u_{-i})).$$

Pareto-efficiency states that the rule always assigns a Pareto-efficient allocation.

Definition 2. A rule f satisfies **Pareto-efficiency** if for any $u \in \mathcal{U}^n$, it holds that there exists no $x \in X$ such that for any $i \in N$, we have

$$u_i(x) \geq u_i(f(u)),$$

and for some $i \in N$, we have

$$u_i(x) > u_i(f(u)).$$

Individual rationality states that the rule always assigns an allocation at least as well as the initial income distribution (endowment).

Definition 3. A rule f satisfies **individual rationality** if for any $u \in \mathcal{U}^n$ and any $i \in N$, it holds that

$$u_i(f(u)) \geq u_i(e).$$

3.2 Envy-freeness for recipients

To define the next property, we need some notation. Given $i, k \in N$. Let $\pi^{ik} : X \rightarrow X$ be such that

$$\pi_i^{ik}(x) = x_k \text{ and } \pi_k^{ik}(x) = x_i,$$

and for any $j \neq i, k$,

$$\pi_j^{ik}(x) = x_j.$$

That is, $\pi^{ik}(x)$ is the allocation in which the assignments of agent i and k at x are exchanged. In other words, the positions of agents i and k are exchanged.

Envy-freeness for recipients states that no agent prefers any recipient's position to his one.

Definition 4. A rule f satisfies **envy-freeness for recipients** if for any $u \in \mathcal{U}^n$ and any $i, k \in N$ such that $f_k(u) > e_k$, it holds that

$$u_i(f(u)) \geq u_i(\pi^{ik}(f(u))).$$

3.3 Envy-freeness on non-negative net trade

Given $i, k \in N$. Let $\tau^{ik} : X \rightarrow X$ be such that

$$\tau_i^{ik}(x) = e_i + x_k - e_k \text{ and } \tau_k^{ik}(x) = e_k + x_i - e_i,$$

and for any $j \neq i, k$,

$$\tau_j^{ik}(x) = x_j.$$

That is, $\tau^{ik}(x)$ is the allocation in which the net trades of agent i and k at x are exchanged. In other words, the positions of agents i and k are exchanged in the view of the net trades.

Envy-freeness on non-negative net trade states that no agent prefers any agent's position in the view of the non-negative net trades to his one.

Definition 5. A rule f satisfies **envy-freeness on non-negative net trade** if for any $u \in \mathcal{U}^n$ and any $i, k \in N$ such that $f_i(u) \geq e_i$ and $f_k(u) \geq e_k$, it holds that

$$u_i(f(u)) \geq u_i(\tau^{ik}(f(u))).$$

4 Rules

We define two rules satisfying *strategy-proofness*, *Pareto-efficiency*, and *individual rationality*. To do so, we need some notation. Let $b(u) \in N$ be such that for any $i \leq b(u)$, $\beta_i \geq \frac{n-1}{n}$ and $\beta_{b(u)+1} < \frac{n-1}{n}$. Define also

$$B(u) \equiv \{i \in N : i \leq b(u)\}.$$

Roughly speaking, $B(u)$ is the sequence of agents who are rich at initial income and have a high value of β_i . In the following rules, the agents in $B(u)$ redistribute some part of their income to poor agents.

4.1 Gross uniform redistribution rule G

Definition 6. The **gross uniform redistribution rule** G is defined as follows: for any $u \in \mathcal{U}^n$ and any $i \in N$,

$$G_i(u) = \begin{cases} G_{b(u)+1} & \text{if } i \in B(u), \\ \min\{e_i, \lambda\} & \text{if } i \notin B(u), \end{cases}$$

where $\lambda \in \mathbb{R}_+$ solves $\sum_{j \in N} G_j(u) = 1$.

Example 1. Let $n = 5$ and $e = (\frac{31}{100}, \frac{26}{100}, \frac{21}{100}, \frac{16}{100}, \frac{6}{100})$. Let $u \in \mathcal{U}^n$.

$$\begin{aligned} B(u) = \emptyset &\Rightarrow G(u) = e, \\ B(u) = \{1\} &\Rightarrow G(u) = \left(\frac{26}{100}, \frac{26}{100}, \frac{21}{100}, \frac{16}{100}, \frac{11}{100}\right), \\ B(u) = \{1, 2\} &\Rightarrow G(u) = \left(\frac{21}{100}, \frac{21}{100}, \frac{21}{100}, \frac{18.5}{100}, \frac{18.5}{100}\right), \\ B(u) = \{1, 2, 3\} &\Rightarrow G(u) = \left(\frac{20}{100}, \frac{20}{100}, \frac{20}{100}, \frac{20}{100}, \frac{20}{100}\right). \end{aligned}$$

4.2 Net uniform redistribution rule T

Definition 7. The **net uniform redistribution rule** T is defined as follows: for any $u \in \mathcal{U}^n$ and any $i \in N$,

$$T_i(u) = \begin{cases} T_{b(u)+1} & \text{if } i \in B(u), \\ e_i + \lambda & \text{if } i \notin B(u), \end{cases}$$

where $\lambda \in \mathbb{R}_+$ solves $\sum_{j \in N} T_j(u) = 1$.

Example 2. Let $n = 5$ and $e = (\frac{31}{100}, \frac{26}{100}, \frac{21}{100}, \frac{16}{100}, \frac{6}{100})$. Let $u \in \mathcal{U}^n$.

$$\begin{aligned}
B(u) = \emptyset &\Rightarrow T(u) = e, \\
B(u) = \{1\} &\Rightarrow T(u) = \left(\frac{27}{100}, \frac{27}{100}, \frac{22}{100}, \frac{17}{100}, \frac{7}{100}\right), \\
B(u) = \{1, 2\} &\Rightarrow T(u) = \left(\frac{24}{100}, \frac{24}{100}, \frac{24}{100}, \frac{19}{100}, \frac{9}{100}\right), \\
B(u) = \{1, 2, 3\} &\Rightarrow T(u) = \left(\frac{22}{100}, \frac{22}{100}, \frac{22}{100}, \frac{22}{100}, \frac{12}{100}\right), \\
B(u) = \{1, 2, 3, 4\} &\Rightarrow T(u) = \left(\frac{20}{100}, \frac{20}{100}, \frac{20}{100}, \frac{20}{100}, \frac{20}{100}\right).
\end{aligned}$$

5 Results

We state our results. All the proofs are in Appendix. The first result is the characterization of the *gross uniform redistribution rule* G .

Theorem 1. A rule satisfies *strategy-proofness*, *Pareto-efficiency*, *individual rationality*, and *envy-freeness for recipients* if and only if it is the *gross uniform redistribution rule* G .

The second result is the characterization of the *net uniform redistribution rule* T .

Theorem 2. A rule satisfies *strategy-proofness*, *Pareto-efficiency*, and *envy-freeness on non-negative net trade* if and only if it is the *net uniform redistribution rule* T .

6 Appendix

6.1 Preliminaries

Lemma 1. For any $i \in N$, any $u_i \in \mathcal{U}$, and any $x \in X$, it holds that

$$\begin{aligned}
u_i(x) &= x_i(1 - \beta_i \frac{n}{n-1}) - (\alpha_i + \beta_i) \frac{1}{n-1} \sum_{j \in L_i(x)} (x_j - x_i) + \beta_i \frac{1}{n-1} \\
&= x_i(1 - \beta_i) - \beta_i \frac{1}{n-1} x_i - (\alpha_i + \beta_i) \frac{1}{n-1} \sum_{j \in L_i(x)} (x_j - x_i) + \beta_i \frac{1}{n-1},
\end{aligned}$$

where $L_i(x) \equiv \{j \in N : x_j \geq x_i\}$.

Proof of Lemma 1. Let $i \in N$, $u_i \in \mathcal{U}$, and $x \in X$. By definition, it holds that

$$u_i(x) = x_i - \alpha_i \frac{1}{n-1} \sum_{j \in L_i(x)} (x_j - x_i) - \beta_i \frac{1}{n-1} \sum_{j \notin L_i(x)} (x_i - x_j).$$

Notice that

$$\begin{aligned} \sum_{j \notin L_i(x)} (x_i - x_j) &= (n - \#L_i(x))x_i - \sum_{j \notin L_i(x)} x_j \\ &= (n - \#L_i(x))x_i - (1 - \sum_{j \in L_i(x)} x_j) \\ &= nx_i - \#L_i(x)x_i - 1 + \sum_{j \in L_i(x)} x_j \\ &= nx_i - 1 + \sum_{j \in L_i(x)} (x_j - x_i). \end{aligned}$$

Thus, it follows that

$$\begin{aligned} u_i(x) &= x_i(1 - \beta_i \frac{n}{n-1}) - (\alpha_i + \beta_i) \frac{1}{n-1} \sum_{j \in L_i(x)} (x_j - x_i) + \beta_i \frac{1}{n-1} \\ &= x_i(1 - \beta_i) - \beta_i \frac{1}{n-1} x_i - (\alpha_i + \beta_i) \frac{1}{n-1} \sum_{j \in L_i(x)} (x_j - x_i) + \beta_i \frac{1}{n-1}. \end{aligned}$$

Therefore, Lemma 1 is valid. $\square$

Lemma 2. Let f be a rule satisfying *Pareto-efficiency*. Let $u \in \mathcal{U}^n$. Denote $f(u) = x$. Then, there exists $i \in \arg \max_{j \in N} x_j$ such that $\beta_i < \frac{n-1}{n}$.

Proof of Lemma 2. Denote $M = \arg \max_{j \in N} x_j$ and $m = \#M$. Suppose, to the contrary, that for any $i \in M$, $\beta_i \geq \frac{n-1}{n}$. For sufficiently small $\epsilon > 0$, let $\hat{x} \in X$ be such that for any $i \in N$,

$$\hat{x}_i = \begin{cases} x_i - \epsilon & \text{if } i \in M, \\ x_i + \frac{m \cdot \epsilon}{n-m} & \text{if } i \notin M. \end{cases}$$

Step 1. For any $i \in M$, it holds that $u_i(\hat{x}) \geq u_i(x)$.

Proof of Step 1. Let $i \in M$. Note that

$$\begin{aligned}
u_i(\hat{x}) &= x_i - \epsilon - \beta_i \frac{1}{n-1} \sum_{j \notin M} ((x_i - \epsilon) - (x_j + \frac{m \cdot \epsilon}{n-m})) \\
&= x_i - \epsilon - \beta_i \frac{1}{n-1} \sum_{j \notin M} (x_i - x_j) + \beta_i \frac{1}{n-1} ((n-m)\epsilon + m \cdot \epsilon) \\
&= x_i - \epsilon - \beta_i \frac{1}{n-1} \sum_{j \notin M} (x_i - x_j) + \beta_i \frac{n}{n-1} \epsilon
\end{aligned}$$

and

$$u_i(x) = x_i - \beta_i \frac{1}{n-1} \sum_{j \notin M} (x_i - x_j).$$

Then, it holds that

$$\begin{aligned}
u_i(\hat{x}) - u_i(x) &= -\epsilon + \beta_i \frac{n}{n-1} \epsilon \\
&= \epsilon(-1 + \beta_i \frac{n}{n-1}) \\
&\geq 0,
\end{aligned}$$

because of $\beta_i \geq \frac{n-1}{n}$. □

Step 2. For any $i \notin M$, it holds that $u_i(\hat{x}) > u_i(x)$.

Proof of Step 2. Let $i \notin M$. Note that

$$\begin{aligned}
u_i(\hat{x}) &= x_i + \frac{m \cdot \epsilon}{n-m} - \alpha_i \frac{1}{n-1} \left(\sum_{j \in M} ((x_j - \epsilon) - (x_i + \frac{m \cdot \epsilon}{n-m})) + \sum_{j \in N \setminus M} \max\{x_j - x_i, 0\} \right) \\
&\quad - \beta_i \frac{1}{n-1} \sum_{j \in N} \max\{x_i - x_j, 0\} \\
&= x_i + \frac{m \cdot \epsilon}{n-m} - \alpha_i \frac{1}{n-1} \left(\sum_{j \in M} (x_j - x_i) - \frac{n \cdot m \cdot \epsilon}{n-m} + \sum_{j \in N \setminus M} \max\{x_j - x_i, 0\} \right) \\
&\quad - \beta_i \frac{1}{n-1} \sum_{j \in N} \max\{x_i - x_j, 0\}
\end{aligned}$$

and

$$u_i(x) = x_i - \alpha_i \frac{1}{n-1} \left(\sum_{j \in M} (x_j - x_i) + \sum_{j \in N \setminus M} \max\{x_j - x_i, 0\} \right) - \beta_i \frac{1}{n-1} \sum_{j \in N} \max\{x_i - x_j, 0\}.$$

Then, it holds that

$$u_i(\hat{x}) - u_i(x) = \frac{m \cdot \epsilon}{n - m} + \alpha_i \frac{1}{n - 1} \frac{n \cdot m \cdot \epsilon}{n - m} > 0.$$

□

Step 1 and Step 2 contradict *Pareto-efficiency*. Therefore, Lemma 2 is valid.

□

Lemma 3. Let f be a rule satisfying *envy-freeness for recipients*. Let $u \in \mathcal{U}^n$. Denote $f(u) = x$. If $x_k > e_k$, then $k \in \arg \min_{j \in N} x_j$.

Proof of Lemma 3. Suppose, to the contrary, that there exists $i, k \in N$ such that $x_k > e_k$ and $x_k > x_i$. Denote $\pi^{ik}(x) = \hat{x}$. Since $L_k(x) = L_i(\hat{x})$, by Lemma 1, we have

$$u_i(\hat{x}) = x_k(1 - \beta_i) - \beta_i \frac{1}{n - 1} x_k - (\alpha_i + \beta_i) \frac{1}{n - 1} \sum_{j \in L_k(x)} (x_j - x_k) + \beta_i \frac{1}{n - 1}$$

and

$$u_i(x) = x_i(1 - \beta_i) - \beta_i \frac{1}{n - 1} x_i - (\alpha_i + \beta_i) \frac{1}{n - 1} \sum_{j \in L_i(x)} (x_j - x_i) + \beta_i \frac{1}{n - 1}.$$

Notice that

$$\begin{aligned} \sum_{j \in L_k(x)} (x_j - x_k) &= \sum_{j \in L_k(x)} (x_j - x_k + x_i - x_i) \\ &= \sum_{j \in L_k(x)} (x_j - x_i) - \sum_{j \in L_k(x)} (x_k - x_i) \\ &= \sum_{j \in L_k(x)} (x_j - x_i) - \sum_{j \in L_k(x)} (x_k - x_i) \\ &\quad + \sum_{j \in L_i(x) \setminus L_k(x)} (x_j - x_i) - \sum_{j \in L_i(x) \setminus L_k(x)} (x_j - x_i) \\ &= \sum_{j \in L_i(x)} (x_j - x_i) - \sum_{j \in L_k(x)} (x_k - x_i) - \sum_{j \in L_i(x) \setminus L_k(x)} (x_j - x_i). \end{aligned}$$

Then, it holds that

$$\begin{aligned} u_i(\hat{x}) - u_i(x) &= (x_k - x_i)(1 - \beta_i) - \beta_i \frac{1}{n - 1} (x_k - x_i) \\ &\quad + (\alpha_i + \beta_i) \frac{1}{n - 1} \left(\sum_{j \in L_k(x)} (x_k - x_i) + \sum_{j \in L_i(x) \setminus L_k(x)} (x_j - x_i) \right) \end{aligned}$$

Since $k \in L_k(x) \neq \emptyset$, it follows that

$$-\beta_i \frac{1}{n-1} (x_k - x_i) + \beta_i \frac{1}{n-1} \sum_{j \in L_k(x)} (x_k - x_i) \geq 0.$$

Thus, we have

$$u_i(\hat{x}) - u_i(x) > 0.$$

However, this contradicts *envy-freeness for recipients*. Therefore, Lemma 3 is valid. $\square$

Lemma 4. Let f be a rule satisfying *Pareto-efficiency* and *envy-freeness for recipients*. Let $u \in \mathcal{U}^n$. Denote $f(u) = x$ and $b(u) = b$. If $x_{b+1} \geq e_{b+1}$, then

$$x_{b+1} = \max_{k \in N} x_k.$$

Proof of Lemma 4. Suppose that $x_i > x_{b+1}$. By Lemma 2, it implies $x_j \geq x_i$ for some $j > b+1$. Then, this means that

$$x_j > x_{b+1} \geq e_{b+1} \geq e_j,$$

that is,

$$x_j > e_j \text{ and } j \notin \arg \min_{k \in N} x_k.$$

However, by Lemma 3, this is a contradiction. $\square$

Lemma 5. Let f be a rule satisfying *envy-freeness on non-negative net trade*. Let $u \in \mathcal{U}^n$. Denote $f(u) = x$. Let $i \in N$ be such that $x_i \geq e_i$ and $\beta_i < \frac{n-1}{n}$. Then, for any $k \in N$, we have $x_i - e_i \geq x_k - e_k$.

Proof of Lemma 5. Suppose, to the contrary, that there exists $k \in N$ such that $x_i - e_i < x_k - e_k$. Denote $\tau^{ik}(x) = \hat{x}$. By Lemma 1, we have⁵

$$\begin{aligned} u_i(\hat{x}) &= (e_i + x_k - e_k) \left(1 - \beta_i \frac{n}{n-1}\right) \\ &\quad - (\alpha_i + \beta_i) \frac{1}{n-1} \sum_{j \in L_i(\hat{x})} (\hat{x}_j - (e_i + x_k - e_k)) + \beta_i \frac{1}{n-1} \\ &= (e_i + x_k - e_k) \left(1 - \beta_i \frac{n}{n-1}\right) \\ &\quad - (\alpha_i + \beta_i) \frac{1}{n-1} \sum_{j \in L_i(\hat{x}) \setminus \{i\}} (\hat{x}_j - (e_i + x_k - e_k)) + \beta_i \frac{1}{n-1} \\ &\geq (e_i + x_k - e_k) \left(1 - \beta_i \frac{n}{n-1}\right) \\ &\quad - (\alpha_i + \beta_i) \frac{1}{n-1} \sum_{j \in L_i(\hat{x}) \setminus \{i\}} (x_j - (e_i + x_k - e_k)) + \beta_i \frac{1}{n-1} \end{aligned}$$

⁵Note that for any $j \neq i, k$, $\hat{x}_j = x_j$ and $\hat{x}_k = e_k + (x_i - e_i) < e_k + (x_k - e_k) = x_k$.

and

$$\begin{aligned}
u_i(x) &= x_i(1 - \beta_i \frac{n}{n-1}) - (\alpha_i + \beta_i) \frac{1}{n-1} \sum_{j \in L_i(x)} (x_j - x_i) + \beta_i \frac{1}{n-1} \\
&= x_i(1 - \beta_i \frac{n}{n-1}) - (\alpha_i + \beta_i) \frac{1}{n-1} \sum_{j \in L_i(x) \setminus \{i\}} (x_j - x_i) + \beta_i \frac{1}{n-1} \\
&= (e_i + x_i - e_i)(1 - \beta_i \frac{n}{n-1}) \\
&\quad - (\alpha_i + \beta_i) \frac{1}{n-1} \sum_{j \in L_i(x) \setminus \{i\}} (x_j - (e_i + x_i - e_i)) + \beta_i \frac{1}{n-1}.
\end{aligned}$$

Notice that $L_i(\hat{x}) \setminus \{i\} \subset L_i(x) \setminus \{i\}$, because $\hat{x}_i = e_i + x_k - e_k > e_i + x_i - e_i = x_i$. Then, it holds that

$$\begin{aligned}
u_i(\hat{x}) - u_i(x) &\geq ((x_k - e_k) - (x_i - e_i))(1 - \beta_i \frac{n}{n-1}) \\
&\quad + (\alpha_i + \beta_i) \frac{1}{n-1} \sum_{j \in L_i(\hat{x}) \setminus \{i\}} ((x_k - e_k) - (x_i - e_i)).
\end{aligned}$$

Since $\beta_i < \frac{n-1}{n}$, we have

$$u_i(\hat{x}) - u_i(x) > 0.$$

However, this contradicts *envy-freeness on non-negative net trade*. Therefore, Lemma 5 is valid. $\square$

Lemma 6. Let f be a rule satisfying *Pareto-efficiency* and *envy-freeness on non-negative net trade*. Let $u \in \mathcal{U}^n$. Denote $f(u) = x$ and $b(u) = b$. If $x_{b+1} \geq e_{b+1}$, then

$$x_{b+1} = \max_{k \in N} x_k.$$

Proof of Lemma 6. Suppose that $x_i > x_{b+1}$. By Lemma 2, it implies $x_j \geq x_i$ for some $j > b+1$. Then, this means that

$$x_j > x_{b+1} \geq e_{b+1} \geq e_j,$$

that is,

$$x_j - e_j > x_{b+1} - e_{b+1}.$$

However, by Lemma 5, this is a contradiction. $\square$

6.2 Proof of Theorem 1

IF PART

This is straightforward, so we omit it.

ONLY IF PART

Let f be a rule satisfying *strategy-proofness*, *Pareto-efficiency*, *individual rationality*, and *envy-freeness for recipients*. Let $u \in \mathcal{U}^n$. We show $f(u) = G(u)$ by mathematical induction. Denote $b(u) = b$.

Step 1. When $\#\{i \in N \setminus B(u) : \alpha_i = \beta_i = 0\} + \#\{i \in B(u) : \beta_i = \frac{n-1}{n}\} = n$, it holds that $f(u) = G(u)$.

Proof of Step 1. Let $u \in \mathcal{U}^n$ be such that $\#\{i \in N \setminus B(u) : \alpha_i = \beta_i = 0\} + \#\{i \in B(u) : \beta_i = \frac{n-1}{n}\} = n$. Notice that for any $i \in N \setminus B(u)$, $\alpha_i = \beta_i = 0$, and that for any $i \in B(u)$, $\beta_i = \frac{n-1}{n}$. Denote $f(u) = x$. We show that for any $i \in N$, $x_i = G_i(u)$.

Sub-step 1-1. For any $i \in N \setminus B(u)$, it follows that $x_i \geq e_i$.

Proof of Sub-step 1-1. This is trivial from *individual rationality*. $\square$

Remark 1. By Sub-step 1-1, we have $x_{b+1} \geq e_{b+1}$.

Sub-step 1-2. For any $i \in B(u)$, it follows that $x_i = x_{b+1}$.

Proof of Sub-step 1-2. Let $i \in B(u)$. By Lemma 4 and Remark 1, we have $x_{b+1} \geq x_i$. Suppose that $x_{b+1} > x_i$. Let $\hat{u}_i \in \mathcal{U}$ be such that $\hat{\alpha}_i = \hat{\beta}_i = 0$. Denote $f(\hat{u}_i, u_{-i}) = \hat{x}$. Then, by *individual rationality* and Lemma 4, it holds that

$$\hat{x}_i = \max_{k \in N} \hat{x}_k.$$

By Lemma 1, we have

$$\begin{aligned} u_i(\hat{x}) &= \hat{x}_i(1 - \beta_i \frac{n}{n-1}) + \beta_i \frac{1}{n-1} \\ &= \beta_i \frac{1}{n-1} \end{aligned}$$

and

$$\begin{aligned} u_i(x) &= x_i(1 - \beta_i \frac{n}{n-1}) - (\alpha_i + \beta_i) \frac{1}{n-1} \sum_{j \in L_i(x)} (x_j - x_i) + \beta_i \frac{1}{n-1} \\ &= -(\alpha_i + \beta_i) \frac{1}{n-1} \sum_{j \in L_i(x)} (x_j - x_i) + \beta_i \frac{1}{n-1}. \end{aligned}$$

Then, since $x_{b+1} > x_i$ and $b+1 \in L_i(x)$, we have

$$u_i(\hat{x}) - u_i(x) = (\alpha_i + \beta_i) \frac{1}{n-1} \sum_{j \in L_i(x)} (x_j - x_i) > 0.$$

However, this contradicts *strategy-proofness*. $\square$

Sub-step 1-3. For any $i \in N$ such that $i \leq b+1$, it follows that $x_i = G_i(u)$.

Proof of Sub-step 1-3. Let $i \in N$ be such that $i \leq b+1$. First, suppose that $x_i > G_i(u)$. By Sub-step 1-2, for any $k \leq b+1$, it follows that $x_k = x_i > G_i(u) = G_k(u)$. Then, by feasibility, there exists $j > b+1$ such that $x_j < G_j(u)$. This implies that

$$x_{b+1} > G_{b+1}(u) \geq e_{b+1} \text{ and } x_{b+1} > G_{b+1}(u) \geq G_j(u) > x_j.$$

However, by Lemma 3, this is a contradiction.

Next, suppose that $x_i < G_i(u)$. Then, by Sub-step 1-2 and Remark 1,

$$e_{b+1} \leq x_{b+1} < G_{b+1}(u).$$

When $e_{b+1} \geq \frac{1}{n}$, since $G_{b+1}(u) = e_{b+1}$, this is a contradiction. When $e_{b+1} < \frac{1}{n}$, since $G_{b+1}(u) = \frac{1}{n}$, by Lemma 4 and feasibility, this is also a contradiction. $\square$

Sub-step 1-4. For any $i \in N$ such that $i > b+1$, it follows that $x_i = G_i(u)$.

Proof of Sub-step 1-4. Let $i \in N$ be such that $i > b+1$. Suppose, to the contrary, that $x_i \neq G_i(u)$. By Sub-step 1-3 and feasibility, without loss of generality, we suppose that $x_i > G_i(u)$ and there exists $j > b+1$ such that $x_j < G_j(u)$. Then, by Sub-step 1-1, it follows that

$$G_j(u) > x_j \geq e_j.$$

Since $G_j(u) > e_j$, by *envy-freeness for recipients* of G and Lemma 3, we have

$$G_j(u) = \min_{k \in N} G_k(u).$$

Since $G_i(u) \geq e_i$, it holds that

$$x_i > G_i(u) \geq e_i.$$

These imply that

$$x_i > e_i \text{ and } x_i > G_i(u) \geq G_j(u) > x_j.$$

However, by Lemma 3, this is a contradiction. $\square$

Thus, Step 1 is completed. $\square$

Step 2. Let $m \in N$ be such that $0 \leq m < n$. Assume that when $\#\{i \in N \setminus B(u) : \alpha_i = \beta_i = 0\} + \#\{i \in B(u) : \beta_i = \frac{n-1}{n}\} \geq m + 1$, it holds that $f(u) = G(u)$. Then, when $\#\{i \in N \setminus B(u) : \alpha_i = \beta_i = 0\} + \#\{i \in B(u) : \beta_i = \frac{n-1}{n}\} = m$, it holds that $f(u) = G(u)$.

Proof of Step 2. Let $u \in \mathcal{U}^n$ be such that $\#\{i \in N \setminus B(u) : \alpha_i = \beta_i = 0\} + \#\{i \in B(u) : \beta_i = \frac{n-1}{n}\} = m$. Denote $f(u) = x$. We show that for any $i \in N$, $x_i = G_i(u)$.

Sub-step 2-1. When $\alpha_{b+1} = \beta_{b+1} = 0$, it follows that $x_{b+1} \geq G_{b+1}(u)$.

Proof of Sub-step 2-1. Suppose, to the contrary, that $x_{b+1} < G_{b+1}(u)$. By *individual rationality*, $\alpha_{b+1} = \beta_{b+1} = 0$ implies $x_{b+1} \geq e_{b+1}$.

On the case of $e_{b+1} \geq \frac{1}{n}$, we have $G_{b+1}(u) = e_{b+1}$. These imply that $x_{b+1} \geq G_{b+1}(u)$, which is a contradiction.

On the case of $e_{b+1} < \frac{1}{n}$, we have $G_{b+1}(u) = \frac{1}{n}$, which implies that $x_{b+1} < \frac{1}{n}$. Since $x_{b+1} \geq e_{b+1}$, by Lemma 4, we have $x_{b+1} = \max_{k \in N} x_k$. These imply that $\max_{k \in N} x_k < \frac{1}{n}$, which contradicts feasibility. $\square$

Sub-step 2-2. It follows that $x_{b+1} \geq G_{b+1}(u)$.

Proof of Sub-step 2-2. Suppose, to the contrary, that $x_{b+1} < G_{b+1}(u)$. By Sub-step 2-1, we assume that u_{b+1} is other than $\alpha_{b+1} = \beta_{b+1} = 0$. Let $\hat{u}_{b+1} \in \mathcal{U}$ be such that $\hat{\alpha}_{b+1} = \hat{\beta}_{b+1} = 0$. Denote $f(\hat{u}_{b+1}, u_{-(b+1)}) = \hat{x}$. By the induction hypothesis, we have $\hat{x} = G(\hat{u}_{b+1}, u_{-(b+1)})$. Since $G(u) = G(\hat{u}_{b+1}, u_{-(b+1)})$, it follows that

$$\hat{x}_{b+1} = G_{b+1}(\hat{u}_{b+1}, u_{-(b+1)}) = G_{b+1}(u) > x_{b+1}.$$

By Lemma 1, we have

$$u_{b+1}(\hat{x}) = \hat{x}_{b+1} \left(1 - \beta_{b+1} \frac{n}{n-1}\right) + \beta_{b+1} \frac{1}{n-1}$$

and

$$u_{b+1}(x) = x_{b+1} \left(1 - \beta_{b+1} \frac{n}{n-1}\right) - (\alpha_{b+1} + \beta_{b+1}) \frac{1}{n-1} \sum_{j \in L_{b+1}(x)} (x_j - x_{b+1}) + \beta_{b+1} \frac{1}{n-1}.$$

Then, we have

$$\begin{aligned}
u_{b+1}(\hat{x}) - u_{b+1}(x) &= (\hat{x}_{b+1} - x_{b+1})\left(1 - \beta_{b+1}\frac{n}{n-1}\right) \\
&\quad + (\alpha_{b+1} + \beta_{b+1})\frac{1}{n-1} \sum_{j \in L_{b+1}(x)} (x_j - x_{b+1}) \\
&> 0.
\end{aligned}$$

However, this contradicts *strategy-proofness*. $\square$

Remark 2. By Sub-step 2-2, we have $x_{b+1} \geq e_{b+1}$.

Sub-step 2-3. For any $i \in B(u)$ such that $\beta_i = \frac{n-1}{n}$, it follows that $x_i = x_{b+1}$.

Proof of Sub-step 2-3. The same argument as Proof of Sub-step 1-2 is valid. So, we omit it. $\square$

Sub-step 2-4. For any $i \in B(u)$ such that $\beta_i > \frac{n-1}{n}$, it follows that $x_i \geq G_i(u)$.

Proof of Sub-step 2-4. Let $i \in B(u)$ be such that $\beta_i > \frac{n-1}{n}$. Suppose, to the contrary, that $x_i < G_i(u)$. Let $\hat{u}_i \in \mathcal{U}$ be such that $\hat{\beta}_i = \frac{n-1}{n}$. Denote $f(\hat{u}_i, u_{-i}) = \hat{x}$. By Sub-step 2-2, it holds that

$$x_{b+1} \geq G_{b+1}(u) = G_{b+1}(\hat{u}_i, u_{-i}) = G_i(\hat{u}_i, u_{-i}) = G_i(u) > x_i.$$

Since, by the induction hypothesis, $\hat{x} = G(\hat{u}_i, u_{-i})$, we have

$$x_{b+1} \geq \hat{x}_i > x_i.$$

By Lemma 1, we have

$$u_i(\hat{x}) = \hat{x}_i(1 - \beta_i) - \beta_i \frac{1}{n-1} \hat{x}_i + \beta_i \frac{1}{n-1}$$

and

$$u_i(x) = x_i(1 - \beta_i) - \beta_i \frac{1}{n-1} x_i - (\alpha_i + \beta_i) \frac{1}{n-1} \sum_{j \in L_i(x)} (x_j - x_i) + \beta_i \frac{1}{n-1}.$$

Then, we have

$$u_i(\hat{x}) - u_i(x) = (\hat{x}_i - x_i)(1 - \beta_i) - \beta_i \frac{1}{n-1} (\hat{x}_i - x_i) + (\alpha_i + \beta_i) \frac{1}{n-1} \sum_{j \in L_i(x)} (x_j - x_i).$$

Since $b+1 \in L_i(x)$ and $x_{b+1} \geq \hat{x}_i$, it follows that

$$-\beta_i \frac{1}{n-1}(\hat{x}_i - x_i) + \beta_i \frac{1}{n-1} \sum_{j \in L_i(x)} (x_j - x_i) \geq 0.$$

Thus, it holds that

$$u_i(\hat{x}) - u_i(x) > 0.$$

However, this contradicts *strategy-proofness*. $\square$

Sub-step 2-5. For any $i \in N$ such that $i > b+1$ and $\alpha_i = \beta_i = 0$, it follows that $x_i \geq G_i(u)$.

Proof of Sub-step 2-5. Suppose, to the contrary, that there exists $i \in N$ such that $i > b+1$, $\alpha_i = \beta_i = 0$, and $x_i < G_i(u)$. By *individual rationality*, $\alpha_i = \beta_i = 0$ implies $x_i \geq e_i$, which means that $G_i(u) > e_i$. Then, by *envy-freeness for recipients* of G and Lemma 3, it follows that

$$G_i(u) = \min_{k \in N} G_k(u).$$

Since $x_i < G_i(u)$, by feasibility, there exists $j \in N$ such that $x_j > G_j(u)$. Then, we have

$$x_j > G_j(u) \geq \min_{k \in N} G_k(u) = G_i(u) > x_i.$$

On the case of $j \geq b+1$, we have $G_j(u) \geq e_j$, which implies that $x_j > e_j$. Then, by Lemma 3, we must have $x_j = \min_{k \in N} x_k$. However, since $x_j > x_i$, this is a contradiction.

On the case of $j < b+1$, we have $G_j(u) = G_{b+1}(u)$. By Lemma 2, there exists $k \in \arg \max_{h \in N} x_h$ such that $\beta_k < \frac{n-1}{n}$. Since $\beta_k < \frac{n-1}{n}$, it follows that $k \geq b+1$, which implies that $G_j(u) \geq G_k(u)$. Then, these imply that $x_k \geq x_j > G_j \geq G_k(u)$, which means a contradiction from the former case. $\square$

Sub-step 2-6. For any $i \in N$ such that $i > b+1$, it follows that $x_i \geq G_i(u)$.

Proof of Sub-step 2-6. Suppose, to the contrary, that there exists $i \in N$ such that $i > b+1$ and $x_i < G_i(u)$. Let $i \in N$ be such agent whose number is the smallest. By Sub-step 2-5, we assume that u_i is other than $\alpha_i = \beta_i = 0$. Let $\hat{u}_i \in \mathcal{U}$ be such that $\hat{\alpha}_i = \hat{\beta}_i = 0$. Denote $f(\hat{u}_i, u_{-i}) = \hat{x}$. By the induction hypothesis, we have $\hat{x} = G(\hat{u}_i, u_{-i})$. Since $G(u) = G(\hat{u}_i, u_{-i})$, it follows that

$$\hat{x}_i = G_i(\hat{u}_i, u_{-i}) = G_i(u) > x_i.$$

We claim that for any $j < i$, $x_j > x_i$ and $x_j \geq \hat{x}_j$. By Sub-step 2-2, 2-3, and 2-4, for any $j \leq b+1$, it holds that

$$x_j \geq G_{b+1}(u).$$

Then, since $G_{b+1}(u) \geq G_i(u) > x_i$ and $G_{b+1}(u) = G_{b+1}(\hat{u}_i, u_{-i}) = G_j(\hat{u}_i, u_{-i}) = \hat{x}_j$, it follows that

$$x_j > x_i \text{ and } x_j \geq \hat{x}_j.$$

By the way of choice i , for any $k \in N$ such that $b+1 < k < i$, it holds that

$$x_k \geq G_k(u).$$

Then, since $G_k(u) \geq G_i(u) > x_i$ and $G_k(u) = G_k(\hat{u}_i, u_{-i}) = \hat{x}_k$, it follows that

$$x_k > x_i \text{ and } x_k \geq \hat{x}_k.$$

Hence, for any $j < i$, we have $x_j > x_i$ and $x_j \geq \hat{x}_j$

By Lemma 1, we have

$$u_i(\hat{x}) = \hat{x}_i(1 - \beta_i) - \beta_i \frac{1}{n-1} \hat{x}_i - (\alpha_i + \beta_i) \frac{1}{n-1} \sum_{j \in \{1, \dots, i-1\}} (\hat{x}_j - \hat{x}_i) + \beta_i \frac{1}{n-1}$$

and

$$u_i(x) = x_i(1 - \beta_i) - \beta_i \frac{1}{n-1} x_i - (\alpha_i + \beta_i) \frac{1}{n-1} \sum_{j \in L_i(x)} (x_j - x_i) + \beta_i \frac{1}{n-1}.$$

Note that $\{1, \dots, i-1\} \subset L_i(x)$, because for any $j < i$, $x_j > x_i$. Then, we have

$$\begin{aligned} u_i(\hat{x}) - u_i(x) &= (\hat{x}_i - x_i)(1 - \beta_i) - \beta_i \frac{1}{n-1} (\hat{x}_i - x_i) \\ &\quad - (\alpha_i + \beta_i) \frac{1}{n-1} \sum_{j \in \{1, \dots, i-1\}} (\hat{x}_j - \hat{x}_i) + (\alpha_i + \beta_i) \frac{1}{n-1} \sum_{j \in L_i(x)} (x_j - x_i) \\ &= (\hat{x}_i - x_i)(1 - \beta_i) - \beta_i \frac{1}{n-1} (\hat{x}_i - x_i) \\ &\quad - (\alpha_i + \beta_i) \frac{1}{n-1} \sum_{j \in \{2, \dots, i-1\}} (\hat{x}_j - \hat{x}_i) + (\alpha_i + \beta_i) \frac{1}{n-1} \sum_{j \in L_i(x) \setminus \{1\}} (x_j - x_i) \\ &\quad - (\alpha_i + \beta_i) \frac{1}{n-1} (\hat{x}_1 - \hat{x}_i) + (\alpha_i + \beta_i) \frac{1}{n-1} (x_1 - x_i). \end{aligned}$$

Since, for any $j < i$, $x_j \geq \hat{x}_j$, it follows that

$$- (\alpha_i + \beta_i) \frac{1}{n-1} \sum_{j \in \{2, \dots, i-1\}} (\hat{x}_j - \hat{x}_i) + (\alpha_i + \beta_i) \frac{1}{n-1} \sum_{j \in L_i(x) \setminus \{1\}} (x_j - x_i) \geq 0.$$

Since we have

$$(\hat{x}_i - x_i)(1 - \beta_i) - \alpha_i \frac{1}{n-1}(\hat{x}_1 - \hat{x}_i) + \alpha_i \frac{1}{n-1}(x_1 - x_i) > 0$$

and

$$-\beta_i \frac{1}{n-1}(\hat{x}_i - x_i) - \beta_i \frac{1}{n-1}(\hat{x}_1 - \hat{x}_i) + \beta_i \frac{1}{n-1}(x_1 - x_i) \geq 0,$$

it holds that

$$u_i(\hat{x}) - u_i(x) > 0.$$

However, this contradicts *strategy-proofness*. □

Sub-step 2-7. For any $i \in N$, it follows that $x_i = G_i(u)$.

Proof of Sub-step 2-7. By Sub-step 2-2 and 2-3, for any $i \in B(u)$ such that $\beta_i = \frac{n-1}{n}$, it follows that

$$x_i = x_{b+1} \geq G_{b+1}(u) = G_i(u).$$

By Sub-step 2-4, for any $i \in B(u)$ such that $\beta_i > \frac{n-1}{n}$, it also follows that

$$x_i \geq G_i(u).$$

By Sub-step 2-6, for any $i \in N$ such that $i > b+1$, it holds that

$$x_i \geq G_i(u).$$

Since, for any $i \in N$, we have $x_i \geq G_i(u)$, by feasibility, we have the desired result. □

Thus, Step 2 is completed. □

Therefore, Theorem 1 is valid. □

6.3 Proof of Theorem 2

IF PART

This is straightforward, so we omit it.

ONLY IF PART

Let f be a rule satisfying *strategy-proofness*, *Pareto-efficiency*, *individual rationality*, and *envy-freeness on non-negative net trade*. Let $u \in \mathcal{U}^n$. We show $f(u) = T(u)$ by mathematical induction. Denote $b(u) = b$.

Step 1. When $\#\{i \in N \setminus B(u) : \alpha_i = \beta_i = 0\} + \#\{i \in B(u) : \beta_i = \frac{n-1}{n}\} = n$, it holds that $f(u) = T(u)$.

Proof of Step 1. Let $u \in \mathcal{U}^n$ be such that $\#\{i \in N \setminus B(u) : \alpha_i = \beta_i = 0\} + \#\{i \in B(u) : \beta_i = \frac{n-1}{n}\} = n$. Notice that for any $i \in N \setminus B(u)$, $\alpha_i = \beta_i = 0$, and that for any $i \in B(u)$, $\beta_i = \frac{n-1}{n}$. Denote $f(u) = x$. We show that for any $i \in N$, $x_i = T_i(u)$.

Sub-step 1-1. For any $i \in N \setminus B(u)$, it follows that $x_i \geq e_i$.

Proof of Sub-step 1-1. This is trivial from *individual rationality*. $\square$

Remark 3. By Sub-step 1-1, we have $x_{b+1} \geq e_{b+1}$.

Sub-step 1-2. For any $i, j \in N \setminus B(u)$, it follows that $x_i - e_i = x_j - e_j$.

Proof of Sub-step 1-2. This is trivial from Lemma 5 and Sub-step 1-1. $\square$

Sub-step 1-3. For any $i \in B(u)$, it follows that $x_i = x_{b+1}$.

Proof of Sub-step 1-3. Let $i \in B(u)$. By Lemma 6 and Remark 3, we have $x_{b+1} \geq x_i$. Suppose that $x_{b+1} > x_i$. Let $\hat{u}_i \in \mathcal{U}$ be such that $\hat{\alpha}_i = \hat{\beta}_i = 0$. Denote $f(\hat{u}_i, u_{-i}) = \hat{x}$. Then, by *individual rationality* and Lemma 6, it holds that

$$\hat{x}_i = \max_{k \in N} \hat{x}_k.$$

By Lemma 1, we have

$$\begin{aligned} u_i(\hat{x}) &= \hat{x}_i(1 - \beta_i \frac{n}{n-1}) + \beta_i \frac{1}{n-1} \\ &= \beta_i \frac{1}{n-1} \end{aligned}$$

and

$$\begin{aligned} u_i(x) &= x_i(1 - \beta_i \frac{n}{n-1}) - (\alpha_i + \beta_i) \frac{1}{n-1} \sum_{j \in L_i(x)} (x_j - x_i) + \beta_i \frac{1}{n-1} \\ &= -(\alpha_i + \beta_i) \frac{1}{n-1} \sum_{j \in L_i(x)} (x_j - x_i) + \beta_i \frac{1}{n-1}. \end{aligned}$$

Then, since $x_{b+1} > x_i$ and $b+1 \in L_i(x)$, we have

$$u_i(\hat{x}) - u_i(x) = (\alpha_i + \beta_i) \frac{1}{n-1} \sum_{j \in L_i(x)} (x_j - x_i) > 0.$$

However, this contradicts *strategy-proofness*. $\square$

Sub-step 1-4. For any $i \in N$ such that $i \leq b+1$, it follows that $x_i = T_i(u)$.

Proof of Sub-step 1-4. Let $i \in N$ be such that $i \leq b+1$. First, suppose that $x_i > T_i(u)$. By Sub-step 1-3, for any $k \leq b+1$, it follows that $x_k = x_i > T_i(u) = T_k(u)$. Then, by feasibility, there exists $j > b+1$ such that $x_j < T_j(u)$. This implies that

$$x_{b+1} - e_{b+1} > T_{b+1}(u) - e_{b+1} = T_j(u) - e_j > x_j - e_j.$$

However, by Sub-step 1-2, this is a contradiction.

Next, suppose that $x_i < T_i(u)$. Then, by feasibility, there exists $j > b+1$ such that $x_j > T_j(u)$. This implies that

$$x_{b+1} - e_{b+1} < T_{b+1}(u) - e_{b+1} = T_j(u) - e_j < x_j - e_j.$$

However, by Sub-step 1-2, this is a contradiction. □

Sub-step 1-5. For any $i \in N$ such that $i > b+1$, it follows that $x_i = T_i(u)$.

Proof of Sub-step 1-5. Let $i \in N$ be such that $i > b+1$. By Sub-step 1-2, it holds that

$$x_i - e_i = x_{b+1} - e_{b+1}.$$

Since, by Sub-step 1-4, $x_{b+1} = T_{b+1}(u)$, it follows that

$$x_i - e_i = x_{b+1} - e_{b+1} = T_{b+1}(u) - e_{b+1} = T_i(u) - e_i,$$

that is, $x_i = T_i(u)$. □

Thus, Step 1 is completed. □

Step 2. Let $m \in N$ be such that $0 \leq m < n$. Assume that when $\#\{i \in N \setminus B(u) : \alpha_i = \beta_i = 0\} + \#\{i \in B(u) : \beta_i = \frac{n-1}{n}\} \geq m+1$, it holds that $f(u) = T(u)$. Then, when $\#\{i \in N \setminus B(u) : \alpha_i = \beta_i = 0\} + \#\{i \in B(u) : \beta_i = \frac{n-1}{n}\} = m$, it holds that $f(u) = T(u)$.

Proof of Step 2. Let $u \in \mathcal{U}^n$ be such that $\#\{i \in N \setminus B(u) : \alpha_i = \beta_i = 0\} + \#\{i \in B(u) : \beta_i = \frac{n-1}{n}\} = m$. Denote $f(u) = x$. We show that for any $i \in N$, $x_i = T_i(u)$.

Sub-step 2-1. When $\alpha_{b+1} = \beta_{b+1} = 0$, it follows that $x_{b+1} \geq T_{b+1}(u)$.

Proof of Sub-step 2-1. Suppose, to the contrary, that $x_{b+1} < T_{b+1}(u)$. By *individual rationality*, $\alpha_{b+1} = \beta_{b+1} = 0$ implies $x_{b+1} \geq e_{b+1}$. Since $x_{b+1} < T_{b+1}(u)$, by feasibility, for some $i \neq b+1$, we must have $x_i > T_i(u)$.

On the case of $i > b+1$, it implies that

$$x_i - e_i > T_i(u) - e_i = T_{b+1}(u) - e_{b+1} > x_{b+1} - e_{b+1},$$

which is incompatible with Lemma 5.

On the case of $i < b+1$, it implies that

$$x_i > T_i(u) = T_{b+1}(u) > x_{b+1},$$

which is incompatible with Lemma 6. □

Sub-step 2-2. It follows that $x_{b+1} \geq T_{b+1}(u)$.

Proof of Sub-step 2-2. Suppose, to the contrary, that $x_{b+1} < T_{b+1}(u)$. By Sub-step 2-1, we assume that u_{b+1} is other than $\alpha_{b+1} = \beta_{b+1} = 0$. Let $\hat{u}_{b+1} \in \mathcal{U}$ be such that $\hat{\alpha}_{b+1} = \hat{\beta}_{b+1} = 0$. Denote $f(\hat{u}_{b+1}, u_{-(b+1)}) = \hat{x}$. By the induction hypothesis, we have $\hat{x} = T(\hat{u}_{b+1}, u_{-(b+1)})$. Since $T(u) = T(\hat{u}_{b+1}, u_{-(b+1)})$, it follows that

$$\hat{x}_{b+1} = T_{b+1}(\hat{u}_{b+1}, u_{-(b+1)}) = T_{b+1}(u) > x_{b+1}.$$

By Lemma 1, we have

$$u_{b+1}(\hat{x}) = \hat{x}_{b+1}(1 - \beta_{b+1} \frac{n}{n-1}) + \beta_{b+1} \frac{1}{n-1}$$

and

$$u_{b+1}(x) = x_{b+1}(1 - \beta_{b+1} \frac{n}{n-1}) - (\alpha_{b+1} + \beta_{b+1}) \frac{1}{n-1} \sum_{j \in L_{b+1}(x)} (x_j - x_{b+1}) + \beta_{b+1} \frac{1}{n-1}.$$

Then, we have

$$\begin{aligned} u_{b+1}(\hat{x}) - u_{b+1}(x) &= (\hat{x}_{b+1} - x_{b+1})(1 - \beta_{b+1} \frac{n}{n-1}) \\ &\quad + (\alpha_{b+1} + \beta_{b+1}) \frac{1}{n-1} \sum_{j \in L_{b+1}(x)} (x_j - x_{b+1}) \\ &> 0. \end{aligned}$$

However, this contradicts *strategy-proofness*. □

Remark 4. By Sub-step 2-2, we have $x_{b+1} \geq e_{b+1}$.

Sub-step 2-3. For any $i \in B(u)$ such that $\beta_i = \frac{n-1}{n}$, it follows that $x_i = x_{b+1}$.

Proof of Sub-step 2-3. The same argument as Proof of Sub-step 1-3 is valid. So, we omit it. $\square$

Sub-step 2-4. For any $i \in B(u)$ such that $\beta_i > \frac{n-1}{n}$, it follows that $x_i \geq T_i(u)$.

Proof of Sub-step 2-4. Let $i \in B(u)$ be such that $\beta_i > \frac{n-1}{n}$. Suppose, to the contrary, that $x_i < T_i(u)$. Let $\hat{u}_i \in \mathcal{U}$ be such that $\hat{\beta}_i = \frac{n-1}{n}$. Denote $f(\hat{u}_i, u_{-i}) = \hat{x}$. By Sub-step 2-2, it holds that

$$x_{b+1} \geq T_{b+1}(u) = T_{b+1}(\hat{u}_i, u_{-i}) = T_i(\hat{u}_i, u_{-i}) = T_i(u) > x_i.$$

Since, by the induction hypothesis, $\hat{x} = T(\hat{u}_i, u_{-i})$, we have

$$x_{b+1} \geq \hat{x}_i > x_i.$$

By Lemma 1, we have

$$u_i(\hat{x}) = \hat{x}_i(1 - \beta_i) - \beta_i \frac{1}{n-1} \hat{x}_i + \beta_i \frac{1}{n-1}$$

and

$$u_i(x) = x_i(1 - \beta_i) - \beta_i \frac{1}{n-1} x_i - (\alpha_i + \beta_i) \frac{1}{n-1} \sum_{j \in L_i(x)} (x_j - x_i) + \beta_i \frac{1}{n-1}.$$

Then, we have

$$u_i(\hat{x}) - u_i(x) = (\hat{x}_i - x_i)(1 - \beta_i) - \beta_i \frac{1}{n-1} (\hat{x}_i - x_i) + (\alpha_i + \beta_i) \frac{1}{n-1} \sum_{j \in L_i(x)} (x_j - x_i).$$

Since $b+1 \in L_i(x)$ and $x_{b+1} \geq \hat{x}_i$, it follows that

$$-\beta_i \frac{1}{n-1} (\hat{x}_i - x_i) + \beta_i \frac{1}{n-1} \sum_{j \in L_i(x)} (x_j - x_i) \geq 0.$$

Thus, it holds that

$$u_i(\hat{x}) - u_i(x) > 0.$$

However, this contradicts *strategy-proofness*. $\square$

Sub-step 2-5. For any $i \in N$ such that $i > b + 1$ and $\alpha_i = \beta_i = 0$, it follows that $x_i \geq T_i(u)$.

Proof of Sub-step 2-5. Suppose, to the contrary, that there exists $i \in N$ such that $i > b + 1$, $\alpha_i = \beta_i = 0$, and $x_i < T_i(u)$. By *individual rationality*, $\alpha_i = \beta_i = 0$ implies $x_i \geq e_i$. Since, by Sub-step 2-2, $x_{b+1} \geq T_{b+1}(u)$, it holds that

$$x_{b+1} - e_{b+1} \geq T_{b+1}(u) - e_{b+1} = T_i(u) - e_i > x_i - e_i,$$

which is incompatible with Lemma 5. $\square$

Sub-step 2-6. For any $i \in N$ such that $i > b + 1$, it follows that $x_i \geq T_i(u)$.

Proof of Sub-step 2-6. Suppose, to the contrary, that there exists $i \in N$ such that $i > b + 1$ and $x_i < T_i(u)$. By Sub-step 2-5, we assume that u_i is other than $\alpha_i = \beta_i = 0$. Let $i \in N$ be such agent whose number is the smallest. Let $\hat{u}_i \in \mathcal{U}$ be such that $\hat{\alpha}_i = \hat{\beta}_i = 0$. Denote $f(\hat{u}_i, u_{-i}) = \hat{x}$. By the induction hypothesis, we have $\hat{x} = T(\hat{u}_i, u_{-i})$. Since $T(u) = T(\hat{u}_i, u_{-i})$, it follows that

$$\hat{x}_i = T_i(\hat{u}_i, u_{-i}) = T_i(u) > x_i.$$

We claim that for any $j < i$, $x_j > x_i$ and $x_j \geq \hat{x}_j$. By Sub-step 2-2, 2-3, and 2-4, for any $j \leq b + 1$, it holds that

$$x_j \geq T_{b+1}(u).$$

Then, since $T_{b+1}(u) \geq T_i(u) > x_i$ and $T_{b+1}(u) = T_{b+1}(\hat{u}_i, u_{-i}) = T_j(\hat{u}_i, u_{-i}) = \hat{x}_j$, it follows that

$$x_j > x_i \text{ and } x_j \geq \hat{x}_j.$$

By the way of choice i , for any $k \in N$ such that $b + 1 < k < i$, it holds that

$$x_k \geq T_k(u).$$

Then, since $T_k(u) \geq T_i(u) > x_i$ and $T_k(u) = T_k(\hat{u}_i, u_{-i}) = \hat{x}_k$, it follows that

$$x_k > x_i \text{ and } x_k \geq \hat{x}_k.$$

Hence, for any $j < i$, we have $x_j > x_i$ and $x_j \geq \hat{x}_j$

By Lemma 1, we have

$$u_i(\hat{x}) = \hat{x}_i(1 - \beta_i) - \beta_i \frac{1}{n-1} \hat{x}_i - (\alpha_i + \beta_i) \frac{1}{n-1} \sum_{j \in \{1, \dots, i-1\}} (\hat{x}_j - \hat{x}_i) + \beta_i \frac{1}{n-1}$$

and

$$u_i(x) = x_i(1 - \beta_i) - \beta_i \frac{1}{n-1} x_i - (\alpha_i + \beta_i) \frac{1}{n-1} \sum_{j \in L_i(x)} (x_j - x_i) + \beta_i \frac{1}{n-1}.$$

Note that $\{1, \dots, i-1\} \subset L_i(x)$, because for any $j < i$, $x_j > x_i$. Then, we have

$$\begin{aligned} u_i(\hat{x}) - u_i(x) &= (\hat{x}_i - x_i)(1 - \beta_i) - \beta_i \frac{1}{n-1} (\hat{x}_i - x_i) \\ &\quad - (\alpha_i + \beta_i) \frac{1}{n-1} \sum_{j \in \{1, \dots, i-1\}} (\hat{x}_j - \hat{x}_i) + (\alpha_i + \beta_i) \frac{1}{n-1} \sum_{j \in L_i(x)} (x_j - x_i) \\ &= (\hat{x}_i - x_i)(1 - \beta_i) - \beta_i \frac{1}{n-1} (\hat{x}_i - x_i) \\ &\quad - (\alpha_i + \beta_i) \frac{1}{n-1} \sum_{j \in \{2, \dots, i-1\}} (\hat{x}_j - \hat{x}_i) + (\alpha_i + \beta_i) \frac{1}{n-1} \sum_{j \in L_i(x) \setminus \{1\}} (x_j - x_i) \\ &\quad - (\alpha_i + \beta_i) \frac{1}{n-1} (\hat{x}_1 - \hat{x}_i) + (\alpha_i + \beta_i) \frac{1}{n-1} (x_1 - x_i). \end{aligned}$$

Since, for any $j < i$, $x_j \geq \hat{x}_j$, it follows that

$$- (\alpha_i + \beta_i) \frac{1}{n-1} \sum_{j \in \{2, \dots, i-1\}} (\hat{x}_j - \hat{x}_i) + (\alpha_i + \beta_i) \frac{1}{n-1} \sum_{j \in L_i(x) \setminus \{1\}} (x_j - x_i) \geq 0.$$

Since we have

$$(\hat{x}_i - x_i)(1 - \beta_i) - \alpha_i \frac{1}{n-1} (\hat{x}_1 - \hat{x}_i) + \alpha_i \frac{1}{n-1} (x_1 - x_i) > 0$$

and

$$- \beta_i \frac{1}{n-1} (\hat{x}_i - x_i) - \beta_i \frac{1}{n-1} (\hat{x}_1 - \hat{x}_i) + \beta_i \frac{1}{n-1} (x_1 - x_i) \geq 0,$$

it holds that

$$u_i(\hat{x}) - u_i(x) > 0.$$

However, this contradicts *strategy-proofness*. □

Sub-step 2-7. For any $i \in N$, it follows that $x_i = T_i(u)$.

Proof of Sub-step 2-7. By Sub-step 2-2 and 2-3, for any $i \in B(u)$ such that $\beta_i = \frac{n-1}{n}$, it follows that

$$x_i = x_{b+1} \geq T_{b+1}(u) = T_i(u).$$

By Sub-step 2-4, for any $i \in B(u)$ such that $\beta_i > \frac{n-1}{n}$, it also follows that

$$x_i \geq T_i(u).$$

By Sub-step 2-6, for any $i \in N$ such that $i > b + 1$, it holds that

$$x_i \geq T_i(u).$$

Since, for any $i \in N$, we have $x_i \geq T_i(u)$, by feasibility, we have the desired result. $\square$

Thus, Step 2 is completed. $\square$

Therefore, Theorem 2 is valid. $\square$

References

- Fehr, E. and K. Schmidt (1999). “A Theory of Fairness, Competition, and Cooperation,” *The Quarterly Journal of Economics* **114**, 817–868.
- Foley, D. K. (1967). “Resource allocation and the public sector,” *Yale Economic Essays* **7**, 45–98.
- Hashimoto, K. and T. Wakayama (2021). “Fair reallocation in economies with single-peaked preferences,” *International Journal of Game Theory* **50**, 773–785.
- Klaus, B., H. Peters, and T. Storcken (1997). “Reallocation of an infinitely divisible good,” *Economic Theory* **10**, 305–333.
- Schmeidler, D. and K. Vind (1972). “Fair net trades,” *Econometrica* **40**, 637–642.