Osaka University of Economics Working Paper Series
No. 2014-2

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November, 2014

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November 13, 2014

Abstract

We consider an implementation of the discounted Shapley value. We modify the Hart–Mas-Colell model as each player discounts future payoffs and proposes not only an allocation but also a coalition. We show that the discounted Shapley value is supported by any stationary subgame perfect equilibrium of the modified game such that in any subgame, the coalition that consists of all active players immediately forms. We also provide conditions for such a stationary subgame perfect equilibrium to exist.

Keywords: Discounted Shapley value, Coalitional bargaining, Noncooperative support, Efficiency

JEL classification codes: C72, C73, C78

*The author is grateful to Minoru Kitahara, Masahiro Okuno-Fujiwara and Kazuo Yamaguchi for their valuable comments. This work was supported by JSPS KAKENHI Grant Number 23730201.

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1 Introduction

[Joo96] introduced the *α-discounted Shapley value*, which is a value that takes into account both marginalism and egalitarianism. The α-discounted Shapley value of coalitional game (N, v) is the Shapley value of coalitonal game (N, w) such that for any $S \in 2^N$, $w(S) = \alpha^{|N|-|S|} v(S)$, where $\alpha \in [0, 1]$. [Joo96] axiomatized the α-discounted Shapley value by the Hart–Mas-Colell consistency and the α-standard ness.[1] If $\alpha = 1$ ($\alpha = 0$), the discounted Shapley value is the Shapley value (egalitarian value).

[HMC96] gave the Shapley value a noncooperative foundation. [HMC96] presented a noncooperative coalitional bargaining model with nontransferable utilities, in which if a proposal is rejected, the player that offers the proposal becomes inactive with probability $1 - \rho$. [HMC96] showed that the Shapley value is implemented as the expected payoff tuple of any equilibrium in the transferable utility case.

We give the α-discounted Shapley value a noncooperative foundation. We incorporate time discounting into the model of [HMC96], in which each player is supposed to not discount future payoffs, in the transferable utility case. The incorporation of time discounting unifies the models of [HMC96] and [Oka93],[2] in which the equilibrium payoff tuple is the egalitarian value. We show that under the discount factor δ, the $\frac{\delta(1-\rho)}{1-\rho\delta}$-discounted Shapley value is implemented as the expected payoff tuple of any subgame-efficient stationary subgame perfect equilibrium (SSPE), which is an SSPE in which the full coalition (the coalition that consists of all active players) forms without delay in every subgame. We also show that the α-discounted Shapley value approximately coincides with the ex post payoff tuple[3] by any subgame-efficient SSPE when ρ and δ go to unity with $\frac{\delta(1-\rho)}{1-\rho\delta}$ converging to α.

[vdBF10] also gave the α-discounted Shapley value a noncooperative foundation. [vdBF10] incorporated time discounting into the model of [PCW01] (the bidding mechanism). [vdBF10] showed that under the discout factor δ, the δ-discounted Shapley value is implemented as the equilibrium payoff tuple. Since $\frac{\delta(1-\rho)}{1-\rho\delta} < \delta$, the value by the Hart–Mas-Colell implementation in the present paper is closer to the egalitarian value than that by the Pérez-Castrillo–Wettstein inplementation in [vdBF10]. This difference might be because the proposer whose proposal is rejected becomes inactive with a certain probability in [HMC96] and with certainty in [PCW01]. Thus, if the proposer whose proposal is rejected probabilistically becomes inactive in [PCW01], both approaches may implement the same value.

[1] A value ϕ is α-standard if for any two-player coalitional game (N, v) and any distinct $i, j \in N$, $\phi_i(N, v) = \frac{v(N)+\alpha v(\{i\})-\alpha v(\{j\})}{2}$.

[2] [Oka93] is an earlier version of [Oka96]. In [Oka93], when a coalition forms, the game ends; in [Oka96], after a coalition forms, the players that are outside of the formed coalition continue bargaining.

[3] The ex post payoff tuple is the payoff tuple that is calculated provided a proposer was selected.

[Joo96] introduced the *egalitarian Shapley value*, which also takes into account both marginalism and egalitarianism. The egalitarian Shapley value is a convex combination of the Shapley value and the egalitarian value. [vdBFJ13] incorporated breakdown of negotiation into the first round of the bidding mechanism in [PCW01] and showed that the egalitarian Shapley value is implemented as the equilibrium payoff tuple.

While in [HMC96], each proposer proposes only an allocation for the full coalition, in the present paper, each proposer proposes both a coalition and an allocation for the coalition. Thus, in the present paper, a coalition structure is endogenously decided and a subcoalition might form, which leads to the inefficiency under the superadditivity. Then, this paper provides a necessary and sufficient condition for a subgame-efficient SSPE to exist. By this condition, it is shown that there is a subgame-efficient SSPE when ρ and δ go to unity with $\frac{\delta(1-\rho)}{1-\rho\delta}$ converging to α (i) only if for any subgame of the underlying coalitional game, the α-discounted Shapley value of the subgame is in the core of the subgame, and (ii) if for any subgame, the α-discounted Shapley value of the subgame is in the interior of the core of the subgame.

[CGL14] and the present paper have studied the Hart–Mas-Colell implementation of the discounted Shapley value independently of each other. [CGL14] introduced time discounting into the model of [HMC96] and showed the coincidence of the discounted Shapley value and the equilibrium payoff tuple of the model. The main difference between [CGL14] and the present paper is as follows. In [CGL14], each proposer proposes only an allocation for the full coalition, that is, the full coalition formation is assumed; in the present paper, each proposer proposes a coalition as well as an allocation for the coalition, and a condition for the full coalition formation is derived. In [CGL14], the nontransferable utility case is also considered; in the present paper, only the transferable utility case is considered.

The remainder of this paper is organized as follows. Section 2 describes a noncooperative coalitional bargaining model; Section 3 shows that the α-discounted Shapley value is supported by any subgame-efficient SSPE; Section 4 gives conditions for a subgame-efficient SSPE to exist; Section 5 concludes this paper. The appendix contains proofs for all propositions.

2 Model

A *coalitional game* is a pair (N, v) such that N is a nonempty finite set and v is a map from 2^N to $\mathbb{R}$ such that $v(\emptyset) = 0$. A coalitional game (N, v) is *superadditive* if for any disjoint $S, T \in 2^N$, $v(S \cup T) \geq v(S) + v(T)$.

Definition 1. For any coalitional game (N, v), the *α-discounted Shapley value of*

(N, v) is $x \in \mathbb{R}^N$ such that for any $i \in N$,

$$x_i = \sum_{i \in S \in 2^N} \frac{(|S|-1)!\,(|N|-|S|)!}{|N|!} \left(\alpha^{|N|-|S|} v(S) - \alpha^{|N|-|S \setminus \{i\}|} v(S \setminus \{i\})\right).$$

The α-discounted Shapley value of (N, v) is the Shapley value of the coalitional game (N, w) such that for any $S \in 2^N$, $w(S) = \alpha^{|N|-|S|} v(S)$.[4] If $\alpha = 1$ ($\alpha = 0$), the α-discounted Shapley value is the Shapley value (egalitarian value).

Let (N, v) be a superadditive coalitional game. For any $(\rho, \delta) \in [0,1]^2 \setminus \{(1,1)\} =: \Pi$, define an extensive game $G(\rho, \delta)$. A *state* is a nonempty subset of N. Let $\mathcal{S} := 2^N \setminus \{\emptyset\}$. In the game, there are infinite rounds, which are classified by states. The structure of the game is as follows. The game begins with a round with state N. At a round with state S, bargaining proceeds as follows:

(i) A player $i \in S$ is selected as a proposer with probability $\frac{1}{|S|}$.

(ii) Player i proposes a pair of a coalition including i and a feasible allocation for the coalition, i.e., (C, x) such that $C \in 2^S \setminus \{\emptyset\}$, $C \ni i$, $x \in \mathbb{R}^C_+$ and $\sum_{j \in C} x_j = v(C)$.

(iii) Each player in $C \setminus \{i\}$ announces her acceptance or rejection of the proposal sequentially according to some predetermined order.

Then, the game proceeds as follows:

- If all players in $C \setminus \{i\}$ accept the proposal, the game ends.
- If a player rejects the proposal,
  - with probability ρ, the game goes to the next round with state S.
  - with probability $1 - \rho$,
    - if $|S| > 1$, it goes to the next round with state $S \setminus \{i\}$.
    - otherwise, the game ends.

If a proposal (C, x) such that $C \ni i$ is accepted at the tth round, player i obtains a payoff of $\delta^{t-1} x_i$; if a proposal (C, x) such that $C \not\ni i$ is accepted or rejection is infinitely repeated, she obtains nothing. δ is the common discount factor.

In this paper, we consider pure strategies. The equilibrium concept employed in the paper is the stationary subgame perfect equilibrium (SSPE), which is the subgame perfect equilibrium such that each player takes the same actions at all rounds with the same state.

For any $(\rho, \delta) \in \Pi$, a strategy tuple σ of $G(\rho, \delta)$ is *subgame efficient* if at every round with state S, every proposal in σ is immediately accepted at the round in σ (*no delay*) and every player proposes full coalition S in σ (*full-coalition formation*).

[4] By using this discounted game (N, w), we can define α-discounted solutions corresponding to other solutions of coalitional games.

This definition is based on [Oka96]. Obviously, $\rho < 1$ or $\delta < 1$ and the superadditivity necessitate no delay and the full-coalition formation for the subgame efficiency, respectively.

For any $S \in \mathcal{S}$, let v^S be the restriction of v to S. For any $S \in \mathcal{S}$ and any $\alpha \in [0,1]$, let $\hat{\phi}^S(\alpha)$ be the α-discounted Shapley value of (S, v^S). For any $(\rho, \delta) \in \Pi$, let $\alpha(\rho, \delta) := \frac{\delta(1-\rho)}{1-\rho\delta}$ and $\phi^S(\rho, \delta) := \hat{\phi}^S(\alpha(\rho, \delta))$. Note that for any $(\rho, \delta) \in \Pi$, $\alpha(\rho, \delta) \in [0,1]$.

3 Equilibrium payoff

Theorem 1 states that the discounted Shapley value is supported as the expected payoff tuple by any subgame-efficient SSPE.

Theorem 1. *Let $(\rho, \delta) \in \Pi$. Let σ be a subgame-efficient SSPE of $G(\rho, \delta)$. Then, the expected payoff tuple by σ in any subgame that begins with any state S is $\phi^S(\rho, \delta)$.*

Remark 1. For any $\bar{\alpha} \in [0,1]$, there exists $(\rho, \delta) \in \Pi$ such that $\alpha(\rho, \delta) = \bar{\alpha}$. For any $\rho \in [0,1)$, $\alpha(\rho, 1) = 1$; for any $\delta \in [0,1)$, $\alpha(1, \delta) = 0$.

Theorem 1 is intuitively explained as follows. At a round with state S, if player i's proposal is accepted, the total surplus for active players is $v(S)$; otherwise, the expected total surplus for active players is $\rho\delta v(S) + (1-\rho)\delta v(S \setminus \{i\})$. As in [HMC96], player i's SSPE payoff is determined according to the difference in these total surpluses, which is $v(S) - (\rho\delta v(S) + (1-\rho)\delta v(S \setminus \{i\})) = (1-\rho\delta)(v(S) - \alpha(\rho, \delta) v(S \setminus \{i\}))$. Thus, the expected SSPE payoff tuple is the $\alpha(\rho, \delta)$-Shapley value. Since the future payoffs are discounted, in the above calculation, $v(S \setminus \{i\})$ is discounted but $v(S)$ is not discounted in the case where the proposal is accepted. Thus, $v(S \setminus \{i\})$ is more discounted than $v(S)$.

Corollary 1 states that the discounted Shapley value is approximately supported as the ex post payoff tuple by any subgame-efficient SSPE.

Corollary 1. *Let $((\rho_n, \delta_n))_{n \in \mathbb{N}}$ be a sequence in Π such that $\lim_{n \to \infty} \rho_n = 1$, $\lim_{n \to \infty} \delta_n = 1$ and $\lim_{n \to \infty} \alpha(\rho_n, \delta_n) = \bar{\alpha}$ for some $\bar{\alpha} \in [0,1]$. Let $(\sigma_n)_{n \in \mathbb{N}}$ be a sequence such that for any $n \in \mathbb{N}$, σ_n is a subgame-efficient SSPE of $G(\rho_n, \delta_n)$. Let $S \in \mathcal{S}$ and $i \in S$. Then, the payoff tuple by σ_n in any subgame that begins with player i's proposing node under state S converges to $\hat{\phi}^S(\bar{\alpha})$ as n goes to infinity.*

Remark 2. For any $\bar{\alpha} \in [0,1]$, there exists a sequence $((\rho_n, \delta_n))_{n \in \mathbb{N}}$ be a sequence in Π such that $\lim_{n \to \infty} \rho_n = 1$, $\lim_{n \to \infty} \delta_n = 1$ and $\lim_{n \to \infty} \alpha(\rho_n, \delta_n) = \bar{\alpha}$. For any sequence $((\rho_n, \delta_n))_{n \in \mathbb{N}}$ in Π such that $\delta_n = 1$ for any $n \in \mathbb{N}$, $\alpha(\rho_n, \delta_n) = 1$ for any $n \in \mathbb{N}$; for any sequence $((\rho_n, \delta_n))_{n \in \mathbb{N}}$ in Π such that $\rho_n = 1$ for any $n \in \mathbb{N}$, $\alpha(\rho_n, \delta_n) = 0$ for any $n \in \mathbb{N}$.

4 Efficiency

Theorem 2 provides a necessary and sufficient condition for subgame efficiency in $G(\rho, \delta)$.

Theorem 2. *Let $(\rho, \delta) \in \Pi$. Then, there exists a subgame-efficient SSPE of $G(\rho, \delta)$ if and only if for any $S, T \in \mathcal{S}$ such that $S \supset T$ and $|S| \geq 2$ and any $i \in T$,*

$$(1-\rho\delta)\left(v(S) - \alpha(\rho,\delta)\, v(S \setminus \{i\}) + \alpha(\rho,\delta) \sum_{j \in T \setminus \{i\}} \phi_j^{S \setminus \{i\}}(\rho,\delta)\right) + \rho\delta \sum_{j \in T} \phi_j^S(\rho,\delta) \geq v(T). \tag{1}$$

Corollary 2 provides the conditions for subgame efficiency when bargaining friction is infinitesimally small.

Corollary 2. *Let $((\rho_n, \delta_n))_{n \in \mathbb{N}}$ be a sequence in Π such that $\lim_{n\to\infty} \rho_n = 1$, $\lim_{n\to\infty} \delta_n = 1$ and $\lim_{n\to\infty} \alpha(\rho_n, \delta_n) = \bar{\alpha}$ for some $\bar{\alpha} \in [0,1]$. Then, for statements (i)–(iii) given below, (i) implies (ii), and (iii) implies (i).*

(i) For some $\bar{n} \in \mathbb{N}$, for any $n \in \mathbb{N}$ such that $n \geq \bar{n}$, there exists a subgame-efficient SSPE of $G(\rho_n, \delta_n)$.

(ii) For any $S \in \mathcal{S}$ such that $|S| \geq 2$, $\hat{\phi}^S(\bar{\alpha})$ is in the core of (S, v^S).

(iii) For any $S \in \mathcal{S}$ such that $|S| \geq 2$, $\hat{\phi}^S(\bar{\alpha})$ is in the interior of the core of (S, v^S).

Remark 3. Suppose that $\bar{\alpha} = 1$. Then, for any $S \in \mathcal{S}$, $\phi^S(\bar{\alpha})$ is the Shapley value of (S, v^S). Thus, if for any $S \in \mathcal{S}$, (S, v^S) is a strictly convex game (i.e., for any $T, U \in 2^S$, $v(T \cup U) + v(T \cap U) > v(T) + v(U)$), then (iii) holds.

(i) $\to$ (ii) in Corollary 2 is intuitively explained as follows. Suppose that there exists a subgame-efficient SSPE σ of $G(\rho_n, \delta_n)$ in the limit $n \to \infty$. Consider a round with any state S. Consider the limit as $n \to \infty$ ($\rho_n \to 1$ and $\delta_n \to 1$). Then, after a rejection at the round, the game goes to the next round with state S without discounting. Thus, by Theorem 1, the expected payoff tuple in the subsequent subgame is the $\bar{\alpha}$-discounted Shapley value $\phi^S(\bar{\alpha})$ of (S, v^S). Hence, any player i's payoff conditional on being a proposer is $v(S) - \sum_{j \in S \setminus \{i\}} \phi_j^S(\bar{\alpha}) = \phi_i^S(\bar{\alpha})$. Suppose that $\phi^S(\bar{\alpha})$ is not in the core of (S, v^S). Then, there exists a coalition $T \subset S$ that blocks $\phi^S(\bar{\alpha})$. Thus, there exists $x \in \mathbb{R}_+^T$ such that for any $j \in T$, $x_j > \phi_j^S(\bar{\alpha})$ and $\sum_{j \in T} x_j = v(T)$. Hence, by the one-shot deviation to proposing (T, x), which is accepted, a player $i \in T$ can improve her payoff from $\phi_i^S(\bar{\alpha})$ to x_i, which is a contradiction.

5 Conclusion

In this paper, we showed that the α-discounted Shapley value is supported by any subgame-efficient SSPE. We also provided conditions for a subgame-efficient SSPE to exist.

Finally, following [MW95], we conjecture that if we allow $(\rho, \delta) = (1, 1)$, (i) when for any $S \in \mathcal{S}$, the core of $\left(S, v^S\right)$ is nonempty, any SSPE payoff tuple in the subgame with active-player set S is in the core of $\left(S, v^S\right)$, and (ii) any allocation in the core of $\left(S, v^S\right)$ is supported as some SSPE payoff tuple in the subgame with active-player set S.

Appendix: Proofs of propositions

Lemma for proof of Theorems 1 and 2

Lemma 1. *For any* $(\rho, \delta) \in \Pi$*, any* $S \in \mathcal{S}$ *such that* $|S| \geq 2$ *and any* $i \in S$*,*

$$\phi_i^S(\rho, \delta) = \frac{1}{|S|}\left(v(S) - \alpha(\rho, \delta)\, v(S \setminus \{i\}) + \alpha(\rho, \delta) \sum_{j \in S \setminus \{i\}} \phi_i^{S \setminus \{j\}}(\rho, \delta)\right).$$

Proof. Since ρ and δ are fixed, we omit (ρ, δ) from $G(\rho, \delta)$, $\alpha(\rho, \delta)$ and $\phi(\rho, \delta)$.

$$\begin{aligned}
&\sum_{j \in S \setminus \{i\}} \phi_i^{S \setminus \{j\}} \\
&= \sum_{j \in S \setminus \{i\}} \sum_{i \in T \in 2^{S \setminus \{j\}}} \frac{(|T|-1)!\,(|S \setminus \{j\}| - |T|)!}{|S \setminus \{j\}|!} \left(\alpha^{|S \setminus \{j\}| - |T|} v(T) - \alpha^{|S \setminus \{j\}| - |T \setminus \{i\}|} v(T \setminus \{i\})\right) \\
&= \sum_{j \in S \setminus \{i\}} \sum_{i \in T \in 2^{S \setminus \{j\}}} \frac{(|T|-1)!\,(|S| - |T| - 1)!}{(|S|-1)!} \left(\alpha^{|S| - |T| - 1} v(T) - \alpha^{|S| - |T \setminus \{i\}| - 1} v(T \setminus \{i\})\right) \\
&= \sum_{j \in S \setminus \{i\}} \sum_{i \in T \in 2^{S} \setminus \{S\}} \mathbf{1}_{j \notin T} \frac{(|T|-1)!\,(|S| - |T| - 1)!}{(|S|-1)!} \left(\alpha^{|S| - |T| - 1} v(T) - \alpha^{|S| - |T \setminus \{i\}| - 1} v(T \setminus \{i\})\right) \\
&= \sum_{i \in T \in 2^{S} \setminus \{S\}} \sum_{j \in S \setminus \{i\}} \mathbf{1}_{j \notin T} \frac{(|T|-1)!\,(|S| - |T| - 1)!}{(|S|-1)!} \left(\alpha^{|S| - |T| - 1} v(T) - \alpha^{|S| - |T \setminus \{i\}| - 1} v(T \setminus \{i\})\right) \\
&= \sum_{i \in T \in 2^{S} \setminus \{S\}} (|S| - |T|) \frac{(|T|-1)!\,(|S| - |T| - 1)!}{(|S|-1)!} \left(\alpha^{|S| - |T| - 1} v(T) - \alpha^{|S| - |T \setminus \{i\}| - 1} v(T \setminus \{i\})\right) \\
&= \sum_{i \in T \in 2^{S} \setminus \{S\}} \frac{(|T|-1)!\,(|S| - |T|)!}{(|S|-1)!} \left(\alpha^{|S| - |T| - 1} v(T) - \alpha^{|S| - |T \setminus \{i\}| - 1} v(T \setminus \{i\})\right).
\end{aligned}$$

Thus,

$$\begin{aligned}
&\frac{1}{|S|}\left(v(S) - \alpha v(S \setminus \{i\}) + \alpha \sum_{j \in S \setminus \{i\}} \phi_i^{S \setminus \{j\}}\right) \\
&= \frac{1}{|S|}\left(v(S) - \alpha v(S \setminus \{i\})\right) + \frac{\alpha}{|S|} \sum_{j \in S \setminus \{i\}} \phi_i^{S \setminus \{j\}} \\
&= \frac{1}{|S|}\left(v(S) - \alpha v(S \setminus \{i\})\right) + \sum_{i \in T \in 2^{S} \setminus \{S\}} \frac{(|T|-1)!\,(|S| - |T|)!}{|S|!} \left(\alpha^{|S| - |T|} v(T) - \alpha^{|S| - |T \setminus \{i\}|} v(T \setminus \{i\})\right) \\
&= \sum_{i \in T \in 2^{S}} \frac{(|T|-1)!\,(|S| - |T|)!}{|S|!} \left(\alpha^{|S| - |T|} v(T) - \alpha^{|S| - |T \setminus \{i\}|} v(T \setminus \{i\})\right) = \phi_i^S.
\end{aligned}$$

Q.E.D.

Proof of Theorem 1 Since ρ and δ are fixed, we omit (ρ, δ) from $G(\rho, \delta)$, $\alpha(\rho, \delta)$ and $\phi(\rho, \delta)$. For any $S \in \mathcal{S}$, let $I^S := \left\{(T, j) \mid T \in 2^S \setminus \{\emptyset\} \wedge j \in T\right\}$ and

$X^S := \mathbb{R}^{I^S}$. For any $S \in \mathcal{S}$, any $x \in X^S$, any $T \in 2^S \setminus \{\emptyset\}$ and any $j \in T$, let $x_j^T := x(T,j)$. For any $S \in \mathcal{S}$, define $f^S : X^S \to X^S$ as for any $x \in X^S$ and any $(T,j) \in I^S$, if $|T| = 1$, $f_j^{ST}(x) = v(T)$, and if $|T| \geq 2$,

$$f_j^{ST}(x) = \frac{1}{|T|}\left(v(T) - \sum_{k \in T\setminus\{j\}} \left(\rho\delta x_k^T + (1-\rho)\delta x_k^{T\setminus\{j\}}\right)\right) + \sum_{k \in T\setminus\{j\}} \frac{1}{|T|}\left(\rho\delta x_j^T + (1-\rho)\delta x_j^{T\setminus\{k\}}\right).$$

By the mathematical induction, show that for any $S \in \mathcal{S}$, $\left(\phi_j^T\right)_{(T,j)\in I^S}$ is a unique fixed point of f^S. Let S be a nonempty subset of N such that $|S| = 1$. For any $x \in X^S$ and any $(T,j) \in I^S$, $f_j^{ST}(x) = v(T) = \phi_j^T$. Thus, $\left(\phi_j^T\right)_{(T,j)\in I^S}$ is a unique fixed point of f^S. Let n be a natural number such that $2 \leq n \leq |N|$. Suppose that for any $S' \in \mathcal{S}$ such that $|S'| = n-1$, $\left(\phi_j^T\right)_{(T,j)\in I^{S'}}$ is a unique fixed point of $f^{S'}$. Let S be a nonempty subset of N such that $|S| = n$.

Lemma 2. *Let x be an element in X^S such that for any $(T,j) \in I^S$ with $T \neq S$, $x_j^T = \phi_j^T$. Then, for any $(T,j) \in I^S$,*

$$f_j^{ST}(x) = \rho\delta x_j^T + (1-\rho\delta)\phi_j^T + \frac{\rho\delta}{|T|}\left(v(T) - \sum_{k\in T} x_k^T\right). \tag{2}$$

Proof. If $|T| = 1$, the both sides of (2) are $v(T)$, and thus, (2) holds. Suppose that $|T| \geq 2$.

$$f_j^{ST}(x) = \frac{1}{|T|}\left(v(T) - \sum_{k \in T\setminus\{j\}} \left(\rho\delta x_k^T + (1-\rho)\delta \phi_k^{T\setminus\{j\}}\right)\right) + \sum_{k \in T\setminus\{j\}} \frac{1}{|T|}\left(\rho\delta x_j^T + (1-\rho)\delta \phi_j^{T\setminus\{k\}}\right).$$

Note that $\sum_{k\in T\setminus\{j\}} \phi_k^{T\setminus\{j\}} = v(T\setminus\{j\})$. Then,

$$f_j^{ST}(x) = \rho\delta x_j^T + \frac{1}{|T|}\left(v(T) - \rho\delta\sum_{k\in T} x_k^T - (1-\rho)\delta v(T\setminus\{j\}) + (1-\rho)\delta \sum_{k\in T\setminus\{j\}} \phi_j^{T\setminus\{k\}}\right).$$

Note that $\alpha = \frac{\delta(1-\rho)}{1-\rho\delta}$. Then,

$$f_j^{ST}(x) = \rho\delta x_j^T + \frac{1-\rho\delta}{|T|}\left(v(T) - \alpha v(T\setminus\{j\}) + \alpha \sum_{k\in T\setminus\{j\}} \phi_j^{T\setminus\{k\}}\right) + \frac{\rho\delta}{|T|}\left(v(T) - \sum_{k\in T} x_k^T\right).$$

Thus, Lemma 1 yields (2). Q.E.D.

By Lemma 2, for any $(T,j) \in I^S$,

$$f_j^{ST}\left(\left(\phi_k^U\right)_{(U,k)\in I^S}\right) = \phi_j^T + \frac{\rho\delta}{|T|}\left(v\left(T\right) - \sum_{k\in T}\phi_k^T\right).$$

Note that $\sum_{k\in T}\phi_k^T = v\left(T\right)$. Then, for any $(T,j)\in I^S$, $f_j^{ST}\left(\left(\phi_k^U\right)_{(U,k)\in I^S}\right) = \phi_j^T$. Thus, $\left(\phi_j^T\right)_{(T,j)\in I^S}$ is a fixed point of f^S. Let x be a fixed point of f^S. Then, for any $i\in S$ and any $(T,j)\in I^{S\backslash\{i\}}$, $f_j^{S\backslash\{i\}T}\left(\left(x_k^U\right)_{(U,k)\in I^{S\backslash\{i\}}}\right) = f_j^{ST}\left(x\right) = x_j^T$. Thus, for any $i\in S$, $\left(x_j^T\right)_{(T,j)\in I^{S\backslash\{i\}}}$ is a fixed point of $f^{S\backslash\{i\}}$. Hence, by the induction hypothesis, for any $i\in S$, $\left(x_j^T\right)_{(T,j)\in I^{S\backslash\{i\}}} = \left(\phi_j^T\right)_{(T,j)\in I^{S\backslash\{i\}}}$. Therefore, for any $(T,j)\in I^S$ with $T\neq S$, $x_j^T = \phi_j^T$. Thus, by Lemma 2, for any $i\in S$,

$$x_i^S = f_i^{SS}\left(x\right) = \rho\delta x_i^S + \left(1-\rho\delta\right)\phi_i^S + \frac{\rho\delta}{|S|}\left(v\left(S\right) - \sum_{k\in S}x_k^S\right). \tag{3}$$

Sum (3) with respect to i over S. Then, since $\sum_{k\in S}\phi_k^S = v\left(S\right)$, $\sum_{k\in S}x_k^S = v\left(S\right)$. Substitute this into (3). Then, for any $i\in S$, $x_i^S = \phi_i^S$. Thus, $x = \left(\phi_j^T\right)_{(T,j)\in I^S}$. Therefore, $\left(\phi_k^T\right)_{(T,k)\in I^S}$ is a unique fixed point of f^S. Hence, by the mathematical induction, for any $S\in\mathcal{S}$, $\left(\phi_j^T\right)_{(T,j)\in I^S}$ is a unique fixed point of f^S.

Let σ be a subgame-efficient SSPE of G. Let u be the element in X^N such that for any $(S,i)\in I^N$, u_i^S is player i's expected payoff by σ at any round with state S. Then, since σ is a subgame-efficient SSPE, for any $(S,i)\in I^N$, if $|S|=1$, $u_i^S = v\left(S\right) = f_i^{NS}\left(u\right)$, and if $|S|\geq 2$,

$$\begin{aligned}u_i^S &= \frac{1}{|S|}\left(v\left(S\right) - \sum_{j\in S\backslash\{i\}}\left(\rho\delta u_j^S + \left(1-\rho\right)\delta u_j^{S\backslash\{i\}}\right)\right) + \sum_{j\in S\backslash\{i\}}\frac{1}{|S|}\left(\rho\delta u_i^S + \left(1-\rho\right)\delta u_i^{S\backslash\{j\}}\right)\\ &= f_i^{NS}\left(u\right).\end{aligned}$$

Thus, u is a fixed point of f^N. Therefore, since $\left(\phi_i^S\right)_{(S,i)\in I^N}$ is a unique fixed point of f^N, $u = \left(\phi_i^S\right)_{(S,i)\in I^N}$. Q.E.D.

Proof of Corollary 1 For any $n\in\mathbb{N}$, by the subgame efficiency of σ_n and Theorem 1, player i's payoff by σ_n in any subgame that starts from player i's proposing node under state S is

$$v\left(S\right) - \sum_{k\in S\backslash\{i\}}\left(\rho_n\delta_n\hat{\phi}_k^S\left(\alpha\left(\rho_n,\delta_n\right)\right) + \left(1-\rho_n\right)\delta_n\hat{\phi}_k^{S\backslash\{i\}}\left(\alpha\left(\rho_n,\delta_n\right)\right)\right),$$

which converges to $v(S)-\sum_{k\in S\setminus\{i\}}\hat{\phi}_k^S(\bar{\alpha})=\hat{\phi}_i^S(\bar{\alpha})$ as n goes to infinity, and that of any player $j\in S\setminus\{i\}$ is

$$\rho_n\delta_n\hat{\phi}_j^S(\alpha(\rho_n,\delta_n))+(1-\rho_n)\,\delta_n\hat{\phi}_j^{S\setminus\{i\}}(\alpha(\rho_n,\delta_n)),$$

which converges to $\hat{\phi}_j^S(\bar{\alpha})$ as n goes to infinity. Q.E.D.

Proof of Theorem 2 Since ρ and δ are fixed, we omit (ρ,δ) from $G(\rho,\delta)$, $\alpha(\rho,\delta)$ and $\phi(\rho,\delta)$.

Necessity Suppose that there exists a subgame-efficient SSPE σ of G. By Theorem 1, For any $S\in\mathcal{S}$ and any $i\in S$, player i's expected payoff by σ at any round with state S is ϕ_i^S. For any $S\in\mathcal{S}$ such that $|S|\geq 2$ and any distinct $i,j\in S$, responder j's expected payoff by rejecting player j's proposal at any round with state S is $\rho\delta\phi_i^S+(1-\rho)\,\delta\phi_i^{S\setminus\{j\}}$. Let S and T be elements in $\mathcal{S}$ such that $S\supset T$ and $|S|\geq 2$. Let $i\in T$. Consider any proposing node of player i at any round with state S. Since σ is a subgame-efficient SSPE, player i's payoff at the proposing node is $v(S)-\sum_{j\in S\setminus\{i\}}\left(\rho\delta\phi_j^S+(1-\rho)\,\delta\phi_j^{S\setminus\{i\}}\right)$. The supremum of player i's payoffs by one-shot deviations to proposals with coalition T to be accepted is $v(T)-\sum_{j\in T\setminus\{i\}}\left(\rho\delta\phi_j^S+(1-\rho)\,\delta\phi_j^{S\setminus\{i\}}\right)$. Since σ is an SPE,

$$v(S)-\sum_{j\in S\setminus\{i\}}\left(\rho\delta\phi_j^S+(1-\rho)\,\delta\phi_j^{S\setminus\{i\}}\right)\geq v(T)-\sum_{j\in T\setminus\{i\}}\left(\rho\delta\phi_j^S+(1-\rho)\,\delta\phi_j^{S\setminus\{i\}}\right).$$

Note that $\sum_{j\in S}\phi_j^S=v(S)$, $\sum_{j\in S\setminus\{i\}}\phi_j^{S\setminus\{i\}}=v(S\setminus\{i\})$ and $\delta(1-\rho)=(1-\rho\delta)\,\alpha$. Then, we obtain (1).

Sufficiency Suppose that for any $S,T\in\mathcal{S}$ such that $S\supset T$ and $|S|\geq 2$ and any $i\in T$, (1) holds. Consider strategy tuple σ such that at any round with any state S, (i) any player $i\in S$ proposes (S,x) such that if $|S|=1$, $x_i=v(S)$, and if $|S|\geq 2$, for any $j\in S\setminus\{i\}$, $x_j=\rho\delta\phi_j^S+(1-\rho)\,\delta\phi_j^{S\setminus\{i\}}$ and (ii) any player $i\in S$ accepts player j's proposal if and only if her share in the proposal is greater than or equal to $\rho\delta\phi_i^S+(1-\rho)\,\delta\phi_i^{S\setminus\{j\}}$. Note that if $|S|\geq 2$,

$$\begin{aligned}x_i&=v(S)-\sum_{j\in S\setminus\{i\}}\left(\rho\delta\phi_j^S+(1-\rho)\,\delta\phi_j^{S\setminus\{i\}}\right)\\&=(1-\delta\rho)\,v(S)-(1-\rho)\,\delta v(S\setminus\{i\})+\rho\delta\phi_i^S\geq\rho\delta\phi_i^S\geq 0.\end{aligned}$$

In σ, any proposal in σ is accepted by all responders. In σ, any player offers a proposal with the full coalition. Thus, σ is subgame efficient. σ is stationary. For any $S\in\mathcal{S}$ and any $i\in S$, let u_i^S be player i's expected payoff by σ at any round with state S. Let S be a nonempty subset of N such that $|S| \geq 2$ and $i \in S$. Since σ involves no delay,

$$u_i^S = \frac{1}{|S|} \left(v(S) - \sum_{j \in S \setminus \{i\}} \left(\rho\delta\phi_j^S + (1-\rho)\,\delta\phi_j^{S\setminus\{i\}} \right) \right) + \sum_{j \in S \setminus \{i\}} \frac{1}{|S|} \left(\rho\delta\phi_i^S + (1-\rho)\,\delta\phi_i^{S\setminus\{j\}} \right).$$

Note that $\sum_{j\in S} \phi_j^S = v(S)$, $\sum_{j\in S\setminus\{i\}} \phi_j^{S\setminus\{i\}} = v(S\setminus\{i\})$ and $\delta(1-\rho) = \alpha(1-\rho\delta)$. Then,

$$\begin{aligned} u_i^S &= \frac{1}{|S|} \left(|S|\,\rho\delta\phi_i^S + (1-\rho\delta)\, v(S) - \delta(1-\rho)\, v(S\setminus\{i\}) + \delta(1-\rho) \sum_{j\in S\setminus\{i\}} \phi_i^{S\setminus\{j\}} \right) \\ &= \rho\delta\phi_i^S + (1-\rho\delta) \frac{1}{|S|} \left((v(S) - \alpha v(S\setminus\{i\})) + \alpha \sum_{j\in S\setminus\{i\}} \phi_i^{S\setminus\{j\}} \right). \end{aligned}$$

Thus, by Lemma 1, $u_i^S = \phi_i^S$. Consider any round with any state S such that $|S| \geq 2$. First, show the unimprovability of responding actions in the round. Any player i's payoff by rejecting any other player j's proposal given other actions in σ is $\rho\delta\phi_i^S + (1-\rho)\,\delta\phi_i^{S\setminus\{j\}}$. Thus, any player's responding actions in σ are unimprovable. Next, consider the unimprovability of proposing actions of any player $i \in S$ at the round. Player i's payoff by σ at her proposing node at the round is $v(S) - \sum_{j\in S\setminus\{i\}} \left(\rho\delta\phi_j^S + (1-\rho)\,\delta\phi_j^{S\setminus\{i\}} \right)$. Consider any one-shot deviation to offering any acceptable proposal with any coalition $T \in 2^S$ with $T \ni i$. Player i's payoff by the deviation is $v(T) - \sum_{j\in T\setminus\{i\}} \left(\rho\delta\phi_j^S + (1-\rho)\,\delta\phi_j^{S\setminus\{i\}} \right)$ at most. Note that $\sum_{j\in S} \phi_j^S = v(S)$, $\sum_{j\in S\setminus\{i\}} \phi_j^{S\setminus\{i\}} = v(S\setminus\{i\})$ and $\delta(1-\rho) = \alpha(1-\rho\delta)$. Then, player i's gain from the deviation is at most

$$v(T) - \rho\delta \sum_{j\in T} \phi_j^S - (1-\rho\delta) \left(v(S) - \alpha v(S\setminus\{i\}) + \alpha \sum_{j\in T\setminus\{i\}} \phi_j^{S\setminus\{i\}} \right).$$

By (1), this is less than or equal to 0. Consider any one-shot deviation to offering any unacceptable proposal. Then, player i's expected payoff by the deviation is $\rho\delta u_i^S$. Thus, player i's gain from the deviation is

$$\begin{aligned} &\rho\delta u_i^S - \left(v(S) - \sum_{j\in S\setminus\{i\}} \left(\rho\delta\phi_j^S + (1-\rho)\,\delta\phi_j^{S\setminus\{i\}} \right) \right) \\ &= -\left((1-\delta\rho)\, v(S) - \delta(1-\rho)\, v(S\setminus\{i\}) \right) \leq 0. \end{aligned}$$

Hence, player i's proposing actions in σ is unimprovable. Therefore, by the one-shot deviation principle, σ is an SPE. Thus, σ is subgame-efficient SSPE. Q.E.D.

Proof of Corollary 2

(i) → (ii) Suppose that (i) holds. Let S be an element in $\mathcal{S}$ such that $|S| \geq 2$. Let $T \in 2^S \setminus \{\emptyset\}$. Then, there exists $i \in T$. Since (i) holds, by Theorem 2, for some $\bar{n} \in \mathbb{N}$, for any $n \in \mathbb{N}$ such that $n \geq \bar{n}$, (1) holds for $(\rho, \delta) = (\rho_n, \delta_n)$. Thus, $\sum_{j \in T} \phi_j^S (\bar{\alpha}) \geq v(T)$. Hence, $\phi^S (\bar{\alpha})$ is in the core of $\left(S, v^S\right)$. Therefore, (ii) holds.

(iii) → (i) Suppose that (iii) holds. Then, for any $S, T \in \mathcal{S}$ such that $S \supset T$ and $|S| \geq 2$ and any $i \in T$, $\sum_{j \in T} \phi_j^S (\bar{\alpha}) > v(T)$, and thus, there exists $\bar{n}_i^{ST} \in \mathbb{N}$ such that for any $n \in \mathbb{N}$ such that $n \geq \bar{n}_i^{ST}$, (1) holds for $(\rho, \delta) = (\rho_n, \delta_n)$. Since $\left\{\bar{n}_i^{ST} \mid i \in T \subset S \in \mathcal{S} \wedge |S| \geq 2\right\}$ is finite, it has the maximum. Let $\bar{n}$ be the maximum. Let $n \in \mathbb{N}$ such that $n \geq \bar{n}$. Then, for any $S, T \in \mathcal{S}$ such that $S \supset T$ and $|S| \geq 2$ and any $i \in T$, since $n \geq \bar{n} \geq \bar{n}_i^{ST}$, (1) holds for $(\rho, \delta) = (\rho_n, \delta_n)$. Thus, by Theorem 2, there exists a subgame-efficient SSPE of $G(\rho_n, \delta_n)$. Therefore, (i) holds. Q.E.D.

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