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**Abstract**

We define a Nash bargaining solution (NBS) for partition function games (PFGs). Based on a PFG, we define an extensive game (EG), which is a propose-respond sequential game where the first rejecter of a proposal exits from the game with a positive probability. We show that the NBS is supported as the expected payoff profile of any stationary subgame perfect equilibrium (SSPE) of the EG such that in any subgame, the coalition of all active players immediately forms. We provide a necessary and sufficient condition for such an SSPE to exist. We also present an example in which delay of agreements occurs.

**Keywords**: Nash bargaining solution, Partition function game, Noncooperative foundation, Rejecter-exit partial breakdown
**JEL classification codes**: C72, C73, C78

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# 1 Introduction

When we regard coalition formation as a bargaining problem (BP), one natural disagreement situation is that if a player disagrees, every player would stand alone. In this situation, each player's threat payoff is her payoff when every player stands alone. Another plausible disagreement situation is that if a player disagrees (or deviates from the agreement), the other players would remain to cooperate and she would be isolated. In this situation, each player's threat payoff is her payoff when she is isolated. When there is no externality in coalition formation, that is, when coalition formation is represented by a characteristic function game (CFG) $(N, v)$, both situations generate the same threat point $(v(\{i\}))_{i \in N}$.

However, if there may be externalities in coalition formation, that is, when coalition formation is represented by a partition function game (PFG) $(N, V)$, the above two situations may generate different threat points. In the situation in which every player stands alone in disagreement, player $i$'s disagreement results in coalition structure $\{\{j\} \mid j \in N\}$, her threat payoff is $V(\{i\}, \{\{j\} \mid j \in N\})$, and thus, the threat point is $(V(\{i\}, \{\{j\} \mid j \in N\}))_{i \in N}$. In the situation in which each disagreer is isolated, player $i$'s disagreement results in coalition structure $\{\{i\}, N \setminus \{i\}\}$, her threat payoff is $V(\{i\}, \{\{i\}, N \setminus \{i\}\})$, and thus, the threat point is $(V(\{i\}, \{\{i\}, N \setminus \{i\}\}))_{i \in N}$. Therefore, in these situations, we define the Nash bargaining solutions (NBSs) of the PFG as the NBSs of the BP such that players split the worth of the grand coalition under the threat points $(V(\{i\}, \{\{j\} \mid j \in N\}))_{i \in N}$ and $(V(\{i\}, N \setminus \{i\}))_{i \in N}$, respectively. We refer to the former and latter as the *fine NBS (fNBS) of PFGs* and *coarse NBS (cNBS) of PFGs*, respectively. The cNBS is newly defined by this work.

There are several remarks on the NBS of PFGs. First, the entries of threat point $(V(\{i\}, \{\{i\}, N \setminus \{i\}\}))_{i \in N}$ are not consistent because for any distinct $i, j \in N$, coalition structures $\{\{i\}, N \setminus \{i\}\}$ and $\{\{j\}, N \setminus \{j\}\}$ do not coexist. However, this inconsistency is reasonable because the players' disagreements and the coalition structures by the disagreements are hypothetical and they do not actually disagree. Another rationale for the inconsistency to be reasonable is to interpret the threat point as the tuple of players' outside options, which do not have to be consistent. Secondly, if positive externalities are strong, the cNBS of PFG does not exist because $\sum_{i \in N} V(\{i\}, \{\{i\}, N \setminus \{i\}\}) > V(N, \{N\})$. Thirdly, if there is no externality, the cNBS and fNBS of PFGs coincide with the NBS of CFGs that are naturally reduced from the PFGs.

According to Gomes (2005), the fNBS of a PFG is supported as the expected payoff profile of a stationary subgame perfect equilibrium (SSPE) of an extensive game (EG).[1] However, the cNBS has not been given any noncooperative foundation.

[1] Okada (2010) investigates EGs based on strategic games. He shows that the strategic-game coun-

Our paper will give the cNBS of PFGs a noncooperative foundation. On the basis of a PFG, we define an EG, a propose-respond sequential bargaining game in which the first rejecter of a proposal exits from the game with a positive probability (*rejecter-exit partial breakdown*). We show that in the limit as the discount factor tends to unity, the expected payoff profile of any full-coalition SSPE (any SSPE such that in any subgame, the coalition of all active players immediately forms) coincides with the cNBS. We also provide a necessary and sufficient condition for a full-coalition SSPE to exist.

We also address delay of agreements. Okada (1996) shows that in his random-proposer model, no SSPE involves delay of agreements, that is, in any SSPE, any player's SSPE proposal is accepted by all responders. This result holds even under externalities. Gomes (2005) and Okada (2010) showed no delay of agreements in their random-proposer models. Not surprisingly, in our model, if the probability of partial breakdown is zero, no SSPE involves delay. However, if the probability is positive, the delay may occur. We present an example in which there exists an SSPE where a player's proposal in the SSPE is rejected by some player.

The fNBS and cNBS are also defined by the following two-step approach: first, define a CFG based on the PFG; secondly, let the NBS of the CFG be the NBS of the PFG. For the fNBS (cNBS, resp.), in the first step, CFG $(N, v)$ based on PFG $(N, V)$ is defined as for any $S \in 2^N \setminus \{\emptyset\}$, $v(S) = V(S, \{S\} \cup \pi)$, where $\pi$ is the finest (coarsest, resp.) partition of $N \setminus S$, that is, $\pi = \{\{i\} \mid i \in N \setminus S\}$ ($\pi = \{N \setminus S \mid i \in N \setminus S\}$,[2] resp.). We refer to this as the *fine way* (*coarse way*, resp.). The approach to define CFGs from PFGs is used to define the Shapley value and core of PFGs. de Clippel and Serrano (2008) and McQuillin (2009) axiomatize the Shapley values of PFGs defined by the fine and coarse ways and call them the *externality-free Shapley value* and the *extended, generalized Shapley value*, respectively. They point out that the externality-free Shapley value and the extended, generalized Shapley value are supported as equilibrium payoff profiles in EGs in Hart and Mas-Colell (1996) and Gul (1989), respectively. Hafalir (2007) defines cores of PFGs by the fine and coarse ways and calls them the *core with singleton expectations* and the *core with merging expectations*, respectively.

In the standard bargaining problem, the disagreement point does not depend on who disagrees (*anonymous disagreement*). On the other hand, in the present paper, the disagreement situation may depend on who disagrees (*nonanonymous disagreement*). Several papers consider BPs with nonanonymous disagreements. Kıbrıs and Tapkı (2010) investigate BPs with nonanonymous disagreements in a cooperative approach. In Kıbrıs and Tapkı (2010), each player's disagreement determines an allocation in disagreement. On the other hand, in the cNBS of our paper, player $i$'s

terpart to the fNBS of PFGs is noncooperatively supported.

[2] If $S \neq N$, $\{N \setminus S \mid i \in N \setminus S\} = \{N \setminus S\}$, and otherwise, $\{N \setminus S \mid i \in N \setminus S\} = \emptyset$.

disagreement determines her payoff and the worth of coalition of the other players but does not determine allocation among the other players, which does not matter in defining the cNBS. Corominas-Bosch (2000) considers a noncooperative bargaining game with nonanonymous disagreements. However, in the model, the number of players is two, and thus, coalition formation is not considered.

A feature of our EGs is the rejecter-exit partial breakdown, in which if players fail to agree, the first rejecter exits from the game with a certain probability. After player $i$ exits from the game by the partial breakdown in the first round, the other players form coalition $N \setminus \{i\}$ in the full-coalition SSPE, coalition structure $\{\{i\}, N \setminus \{i\}\}$ is realized, and player $i$ then obtains a payoff of $V(\{i\}, \{\{i\}, N \setminus \{i\}\})$. This is behind the fact that the expected payoff profile of any full-coalition SSPE in the limit is equal to the cNBS. Miyakawa (2008), Calvo (2008) and Hart and Mas-Colell (1996) consider partial breakdowns. In Miyakawa (2008), a responder is randomly selected and exits from the game. In Calvo (2008), a player is randomly selected and exits from the game. In Hart and Mas-Colell (1996), the proposer exits from the game. In their models, there is no externality in coalition formation. On the other hand, papers studying coalitional bargaining with externalities have not considered partial breakdown (e.g., Bloch (1996) and Ray and Vohra (1999)).

The remainder of the paper is organized as follows: Section 2 defines NBSs of PFGs; Section 3 presents an EG based on a PFG; Section 4 shows that the cNBS is supported by the expected payoff profile of any efficient SSPE in the limit; Section 5 provides a necessary and sufficient condition that there exists an SSPE in which all active players cooperate immediately; Section 6 addresses the delay of agreements in the EG; Section 7 considers mergers of firms in Cournot and Bertrand competitions as applications; and Section 8 concludes the paper. The proofs of all propositions are given in the Appendix.

# 2 Nash bargaining solution

For any function $f$ and any $x$ in the domain of $f$, let $f_x$ be the image of $x$ under $f$, that is, $f_x := f(x)$. For any set $N$, a *partition* of $N$ is $\pi$ such that $\pi \not\ni \emptyset$, $S \cap T = \emptyset$ for any distinct $S, T \in \pi$ and $\bigcup \pi = N$.[3] For any nonempty set $N$, let $\Pi^N$ be the set of partitions of $N$. For any partition $\pi$ of any nonempty set, for any $i \in \bigcup \pi$, let $[i]_\pi$ be the equivalence class of $i$ by $\pi$. For any nonempty set $N$ and any $S \in 2^N \setminus \{\emptyset\}$, let $\underline{\pi}_S^N$ ($\overline{\pi}_S^N$, resp.) be $\{S\} \cup \pi$, where $\pi$ is the finest (coarsest, resp.) partition of $N \setminus S$, that is, $\pi := \{\{i\} \mid i \in N \setminus S\}$ ($\pi := \{N \setminus S \mid i \in N \setminus S\} = \{N \setminus S\} \setminus \{\emptyset\}$, resp.).

[3] According to this definition, the empty set is a unique partition of the empty set.

A *bargaining problem (BP)* is a triple $(N, B, d)$ such that $N$ is a nonempty finite set, $B \subset \mathbb{R}^N$ and $d \in \mathbb{R}^N$. For any BP $(N, B, d)$, a *Nash bargaining solution (NBS)* of $(N, B, d)$ is a solution of $\max_{x \in B} \prod_{i \in N} (x_i - d_i)$ s.t. $x \geq d$. A *characteristic function game (CFG)* is a pair $(N, v)$ such that $N$ is a nonempty finite set and $v$ is a function from $2^N \setminus \{\emptyset\}$ to $\mathbb{R}$. For any CFG $(N, v)$, a *Nash bargaining solution (NBS)* of $(N, v)$ is an NBS of BP $\left(N, \left\{x \in \mathbb{R}^N \mid \sum_{i \in N} x_i \leq v_N\right\}, \left(v_{\{i\}}\right)_{i \in N}\right)$. For any nonempty finite set $N$, let $\mathcal{C}^N := \left\{(S, \pi) \in 2^N \times \Pi^N \mid S \in \pi\right\}$. A *partition function game (PFG)* is a pair $(N, V)$ such that $N$ is a nonempty finite set and $V$ is a function from $\mathcal{C}^N$ to $\mathbb{R}$. For any function $V$ from $\mathcal{C}^N$ to $\mathbb{R}$ for some nonempty finite set $N$, let $\underline{v}^V$ ($\overline{v}^V$, resp.) be the function from $2^N \setminus \{\emptyset\}$ to $\mathbb{R}$ such that for any $S \in 2^N \setminus \{\emptyset\}$, $\underline{v}^V_S = V_{\left(S, \underline{\pi}^N_S\right)}$ ($\overline{v}^V_S = V_{\left(S, \overline{\pi}^N_S\right)}$, resp.).

**Definition 1.** For any PFG $(N, V)$, a *fine Nash bargaining solution (fNBS)* (*coarse Nash bargaining solution (cNBS)*, resp.) of $(N, V)$ is an NBS of CFG $\left(N, \underline{v}^V\right)$ ($\left(N, \overline{v}^V\right)$, resp.).

Behind the fNBS, there is the situation in which if any player disagrees, each player would stand alone. Behind the cNBS, there is the situation in which if a player disagrees, the other players would remain to cooperate and she would be isolated.

Since for any distinct $i, j \in N$, coalition structures $\{\{i\}, N \setminus \{i\}\}$ and $\{\{j\}, N \setminus \{j\}\}$ do not coexist, the entries of threat point $\left(V_{(\{i\}, \{\{i\}, N \setminus \{i\}\})}\right)_{i \in N}$ are not consistent. However, this inconsistency is reasonable because the players' disagreements and the coalition structures by the disagreements are hypothetical and they do not actually disagree. Also, it is reasonable because we can interpret the threat point as the tuple of players' outside options, which do not have to be consistent.

*Remark* 1. There exists an fNBS (cNBS, resp.) if and only if $\sum_{i \in N} V_{\left(\{i\}, \underline{\pi}^N_{\{i\}}\right)} \leq V_{(N, \{N\})}$ ($\sum_{i \in N} V_{\left(\{i\}, \overline{\pi}^N_{\{i\}}\right)} \leq V_{(N, \{N\})}$, resp.). If the grand coalition is efficient, that is, for any $\pi \in \Pi^N$, $V_{(N, \{N\})} \geq \sum_{S \in \pi} V_{(S, \pi)}$, then an fNBS exists, but a cNBS does not necessarily exist.

For any function $v$ from $2^N \setminus \{\emptyset\}$ to $\mathbb{R}$ for some nonempty finite set $N$, let $x^v$ be the function from $N$ to $\mathbb{R}$ such that for any $i \in N$, $x^v_i = \frac{v_N - \sum_{j \in N} v_{\{j\}}}{|N|} + v_{\{i\}}$. For any function $V$ from $\mathcal{C}^N$ to $\mathbb{R}$ for some nonempty finite set $N$, let $\underline{x}^V := x^{\underline{v}^V}$ and $\overline{x}^V := x^{\overline{v}^V}$.

*Remark* 2. If there exists an fNBS (cNBS, resp.), it is unique and is given by $\underline{x}^V$ ($\overline{x}^V$, resp.).

# 3 Extensive games

In the following sections, fix a PFG $(N, V)$. For any $(\delta, p) \in [0,1) \times [0,1]$, define an extensive game $G(\delta, p)$ as follows. A *state* is $\pi \in \bigcup_{S \in 2^N} \Pi^S$. $\pi$ represents a coalition structure of inactive players. For any state $\pi$, let $A^\pi = N \setminus \bigcup \pi$. $A^\pi$ represents the set of active players. Each active player $i$ owns coalition $\{i\}$. In a round with state $\pi$ with $A^\pi \neq \emptyset$, bargaining proceeds as follows. Player $i \in A^\pi$ is selected with probability $\frac{1}{|A^\pi|}$. Player $i$ offers a proposal $(S, x)$ such that $i \in S \in 2^{A^\pi}$, $x \in \mathbb{R}^S$ and $\sum_{j \in S} x_j = 0$ (the proposal means that player $i$ offers monetary term $x_j$ for player $j$'s resource). Each player $j \in S \setminus \{i\}$ announces her acceptance or rejection of the proposal according to a predetermined order until a responder rejects it or all responders accept it. If all responders accept it, the state is updated to $\pi \cup \{S\}$ (the proposer exits from the game with owning coalition $S$ and all the responders exit with no owning coalition). Otherwise, the state remains $\pi$ (the activity of players and the ownership of coalitions do not change) with probability $p$ and is updated to $\pi \cup \{\{j\}\}$ (the responder $j$ exits from the game with owning coalition $\{j\}$) with probability $1 - p$, where $j$ is the rejecter of the proposal. In a round with state $\pi$ with $A^\pi = \emptyset$, no bargaining occurs, and the state remains $\pi$. In each case, the game proceeds to a new round with the updated state. The game starts from a round with state $\emptyset$. In the game, there are four types of players: active players, players who became inactive by rejecting proposals, players who became inactive by their proposals being accepted and players who became inactive by accepting proposals. The first type of player owns her singleton coalition; the second, her singleton coalition; the third, the coalition in the accepted proposal; and the fourth, no coalition. For any complete history $h$, player $i$'s payoff is defined as follows. For any $t \in \mathbb{N}$,[4] let $\rho^t$ be the state in the end of the $t$th round in $h$ and $\pi^t := \rho^t \cup \left\{\{i\} \mid i \in A^{\rho^t}\right\}$. Let $\left(R^t\right)_{t \in \mathbb{Z}_+}$ be the sequence such that $R^0 = \{\{i\} \mid i \in N\}$; for any $t \in \mathbb{N}$, $R^t = R^{t-1} \setminus (S \setminus \{i\})$ if player $i$'s proposal with coalition $S$ is accepted by all responders in the $t$th round in $h$ and $R^t = R^{t-1}$ if the proposal is rejected by some responder in $h$. For any $t \in \mathbb{N}$ and any $i \in N$, let $x_i^t$ be the transfer to player $i$ in the $t$th round in $h$. Then, player $i$'s payoff in $h$ is $\sum_{t \in \mathbb{N}} \delta^{t-1} \left( (1-\delta) \mathbf{1}_{i \in R^t} V_{\left([i]_{\pi^t}, \pi^t\right)} + x_i^t \right)$.

**Definition 2.** A strategy profile $s$ is a *stationary subgame perfect equilibrium (SSPE)* if $s$ is a subgame perfect equilibrium, and in $s$, in any round with the same state, players take the same actions.

**Definition 3.** A strategy profile is a *full-coalition* strategy profile if in the strategy profile, in every subgame starting with state $\pi \neq \emptyset$, coalition $A^\pi$ is immediately formed.

[4] In this paper, $\mathbb{N} \not\ni 0$.

For any state $\pi$ with $A^\pi \neq \emptyset$, let $V^\pi$ be the function from $\mathcal{C}^{A^\pi}$ to $\mathbb{R}$ such that for any $(S,\rho) \in \mathcal{C}^{A^\pi}$, $V^\pi_{(S,\rho)} = V_{(S,\rho\cup\pi)}$. A *subgame* of $(N,V)$ is $(A^\pi, V^\pi)$ such that $\pi$ is a state with $A^\pi \neq \emptyset$. If in any subgame $(M,U)$ of $(N,V)$, the grand coalition is efficient, that is, $U_{(M,\{M\})} \geq \sum_{S\in\pi} U_{(S,\pi)}$ for any $\pi \in \Pi^M$ and this inequality strictly holds for $\pi = \{\{i\} \mid i \in M\}$, then, for any $(\delta,p) \in [0,1)\times[0,1]$, full-coalition strategy profiles coincide with subgame-efficient strategy profiles, that is, strategy profiles that are Pareto efficient in any subgame of $G(\delta,p)$. Subgame-efficiency is defined by Okada (1996). Hafalir (2007) shows that if $(N,V)$ is convex, that is, for any subgame $(M,U)$ of $(N,V)$ and any $S,T \in 2^M \setminus \{\emptyset\}$ with $S\cup T = M$, $U_{(S\cup T,\{S\cup T\})} + U_{(S\cap T,\{S\cap T, S\setminus T, T\setminus S\})} \geq U_{(S,\{S,T\setminus S\})} + U_{(T,\{T,S\setminus T\})}$, then in any subgame of $(N,V)$, the grand coalition is efficient.

For any state $\pi$, let $\underline{v}^\pi := \underline{v}^{V^\pi}$ and $\overline{v}^\pi := \overline{v}^{V^\pi}$.[5] For any state $\pi$, let $\underline{x}^\pi := \underline{x}^{V^\pi}$ and $\overline{x}^\pi := \overline{x}^{V^\pi}$.[6]

# 4 Support for Nash bargaining solution

We show that any full-coalition SSPE brings the same expected payoff profile in any subgame starting with the same state, and we explicitly characterize the expected payoff profile. By this characterization, the limit of the expected payoff profile of any full-coalition SSPE of $G(\delta,p)$ as $\delta$ tends to unity is proved to be the cNBS, and the expected payoff profile of any full-coalition SSPE of $G(\delta,1)$ is proved to be the fNBS.

**Theorem 1.** *Let $(\delta,p) \in [0,1)\times[0,1]$. Let $s$ be a full-coalition SSPE of $G(\delta,p)$. Then, for any state $\pi$, for any $i \in A^\pi$, player $i$'s expected payoff of $s$ in the subgame of $G(\delta,p)$ starting with state $\pi$ is $\frac{1-\delta}{1-p\delta}\underline{x}^\pi_i + \frac{(1-p)\delta}{1-p\delta}\overline{x}^\pi_i$.*

**Corollary 1.** *If there exists a cNBS of $(N,V)$, for any $p \in [0,1)$, the limit of the expected payoff profile of any full-coalition SSPE of $G(\delta,p)$ as $\delta$ tends to unity is equal to the cNBS. If there exists an fNBS of $(N,V)$, for any $\delta \in [0,1)$, the expected payoff profile of any full-coalition SSPE of $G(\delta,1)$ is equal to the fNBS.*

Suppose that $\delta$ is sufficiently close to unity. Then, the following argument approximately holds. In the first round, by rejection, responder $i$ is excluded from the society with probability $1-p$, and in the next round, the coalition of the other players forms in any full-coalition SSPE and responder $i$ obtains a payoff of

[5] By definition, for any state $\pi$ and any $S \in 2^{A^\pi}\setminus\{\emptyset\}$, $\underline{v}^\pi_S = V_{\left(S,\underline{\pi}^{A^\pi}_S\cup\pi\right)}$ and $\overline{v}^\pi_S = V_{\left(S,\overline{\pi}^{A^\pi}_S\cup\pi\right)}$.

[6] By definition, for any state $\pi$ and any $i \in A^\pi$, $\underline{x}^\pi_i = \frac{V_{(A^\pi,\{A^\pi\}\cup\pi)} - \sum_{j\in A^\pi} V_{\left(\{j\},\underline{\pi}^{A^\pi}_{\{j\}}\cup\pi\right)}}{|A^\pi|} + V_{\left(\{i\},\underline{\pi}^{A^\pi}_{\{i\}}\cup\pi\right)}$ and $\overline{x}^\pi_i = \frac{V_{(A^\pi,\{A^\pi\}\cup\pi)} - \sum_{j\in A^\pi} V_{\left(\{j\},\overline{\pi}^{A^\pi}_{\{j\}}\cup\pi\right)}}{|A^\pi|} + V_{\left(\{i\},\overline{\pi}^{A^\pi}_{\{i\}}\cup\pi\right)}$.

$V_{(\{i\},N\setminus\{i\})} = \overline{v}^V_{\{i\}}$. Thus, the threat for player $i$ in the first round is $\overline{v}^V_{\{i\}}$. In the first round, in any full-coalition SSPE, all players cooperate and share $V_{(N,\{N\})} = \overline{v}^V_N$. Therefore, the expected payoff profile of any full-coalition SSPE is the cNBS.

Suppose that $p = 1$. In the first round, by rejection, responder $i$ obtains an instant payoff of $(1-\delta)\, V_{(\{i\},\{\{j\}|j\in N\})} = (1-\delta)\, \underline{v}^V_{\{i\}}$. Thus, the threat for player $i$ is $\underline{v}^V_{\{i\}}$. In the first round, in any full-coalition SSPE, all players cooperate and share $V_{(N,\{N\})} = \underline{v}^V_N$. Therefore, the expected payoff profile of any full-coalition SSPE is the fNBS.

# 5 Conditions for full coalition formation

We provide a necessary and sufficient condition for a full-coalition SSPE to exist.

**Theorem 2.** *Let $(\delta, p) \in [0,1) \times [0,1]$. Then, there exists a full-coalition SSPE of $G(\delta,p)$ if and only if for any state $\pi$ with $A^\pi \neq \emptyset$ and any $S \in 2^{A^\pi} \setminus \{\emptyset\}$,*

$$\frac{\overline{v}^\pi_{A^\pi} - \sum_{k\in A^\pi} \frac{(1-\delta)\underline{v}^\pi_{\{k\}} + \delta(1-p)\overline{v}^\pi_{\{k\}}}{1-\delta p}}{|A^\pi|} \geq \frac{(1-\delta)\,\underline{v}^\pi_S + \delta \overline{v}^\pi_S - \sum_{k\in S} \frac{(1-\delta)\underline{v}^\pi_{\{k\}} + \delta(1-p)\overline{v}^\pi_{\{k\}}}{1-\delta p}}{\delta p\,|S| + (1-\delta p)\,|A^\pi|}, \tag{1}$$

*and for any state $\pi$ with $|A^\pi| \geq 2$ and any distinct $i, j \in A^\pi$,*

$$\begin{aligned}
&\overline{v}^\pi_{A^\pi} - \sum_{k\in A^\pi} \frac{(1-\delta)\,\underline{v}^\pi_{\{k\}} + \delta\,(1-p)\,\overline{v}^\pi_{\{k\}}}{1-\delta p} + \frac{\delta\,(1-p)}{1-\delta p}\overline{v}^\pi_{\{i\}} \\
&\geq \frac{\delta\,(1-p)}{1-\delta p} \frac{\overline{v}^{\pi\cup\{\{j\}\}}_{A^\pi\setminus\{j\}} - \sum_{k\in A^\pi\setminus\{j\}} \frac{(1-\delta)\underline{v}^{\pi\cup\{\{j\}\}}_{\{k\}} + \delta(1-p)\overline{v}^{\pi\cup\{\{j\}\}}_{\{k\}}}{1-\delta p}}{|A|^\pi - 1} \\
&\quad + \frac{\delta\,(1-p)}{1-\delta p} \frac{(1-\delta)\,\underline{v}^{\pi\cup\{\{j\}\}}_{\{i\}} + \delta\,(1-p)\,\overline{v}^{\pi\cup\{\{j\}\}}_{\{i\}}}{1-\delta p}.
\end{aligned} \tag{2}$$

**Corollary 2.** *If for some $\bar{p} \in [0,1)$, for any $p \in [\bar{p}, 1)$, for some $\bar{\delta} \in [0,1)$, for any $\delta \in [\bar{\delta}, 1)$, there exists a full-coalition SSPE of $G(\delta,p)$, then for any state $\pi$ with $A^\pi \neq \emptyset$ and any $S \in 2^{A^\pi} \setminus \{\emptyset\}$,*

$$\frac{\overline{v}^\pi_{A^\pi} - \sum_{k\in A^\pi} \overline{v}^\pi_{\{k\}}}{|A^\pi|} \geq \frac{\overline{v}^\pi_S - \sum_{k\in S} \overline{v}^\pi_{\{k\}}}{|S|}, \tag{3}$$

*and for any state $\pi$ with $|A^\pi| \geq 2$ and any distinct $i, j \in A^\pi$,*

$$\overline{v}^\pi_{A^\pi} - \sum_{k\in A^\pi} \overline{v}^\pi_{\{k\}} + \overline{v}^\pi_{\{i\}} \geq \frac{\overline{v}^{\pi\cup\{\{j\}\}}_{A^\pi\setminus\{j\}} - \sum_{k\in A^\pi\setminus\{j\}} \overline{v}^{\pi\cup\{\{j\}\}}_{\{k\}}}{|A^\pi| - 1} + \overline{v}^{\pi\cup\{\{j\}\}}_{\{i\}}. \tag{4}$$

*If the system of inequalities above strictly holds, then for some $\bar{p} \in [0,1)$, for any*

$p \in [\bar{p}, 1)$, *for some* $\bar{\delta} \in [0,1)$, *for any* $\delta \in \left[\bar{\delta}, 1\right)$, *there exists a full-coalition SSPE of* $G(\delta, p)$. *If for some* $\bar{\delta} \in [0,1)$, *for any* $\delta \in \left[\bar{\delta}, 1\right)$, *there exists a full-coalition SSPE of* $G(\delta, 1)$, *then for any state* $\pi$ *with* $A^\pi \neq \emptyset$ *and any* $S \in 2^{A^\pi} \setminus \{\emptyset\}$,

$$\frac{\overline{v}^\pi_{A^\pi} - \sum_{k \in A^\pi} \underline{v}^\pi_{\{k\}}}{|A^\pi|} \geq \frac{\overline{v}^\pi_S - \sum_{k \in S} \underline{v}^\pi_{\{k\}}}{|S|}. \tag{5}$$

*If the system of inequalities above strictly holds, then for some* $\bar{\delta} \in [0,1)$, *for any* $\delta \in \left[\bar{\delta}, 1\right)$, *there exists a full-coalition SSPE of* $G(\delta, 1)$.

(3) is a condition for any player to offer a proposal with the full coalition in any round with state $\pi$. (4) is a condition for any player to offer a proposal to be accepted in any round with state $\pi$. (5) is a condition for any player to offer a proposal with the full coalition to be accepted in any round with state $\pi$.

*Remark* 3. For any $S \in 2^{A^\pi} \setminus \{\emptyset\}$, (3) ((5), resp.) holds if and only if the NBS of $(A^\pi, \overline{v}^\pi)$ $((A^\pi, \underline{v}^\pi)$, resp.) is in the core of $(A^\pi, \overline{v}^\pi)$.[7]

*Remark* 4. By (3) with $|S| = 1$, if for some $\bar{p} \in [0,1)$, for any $p \in [\bar{p}, 1)$, for some $\bar{\delta} \in [0,1)$, for any $\delta \in \left[\bar{\delta}, 1\right)$, there exists a full-coalition SSPE of $G(\delta, p)$, a cNBS of $(N, v)$ exists, and thus, by Corollary 1, the limit of the expected payoff profile of the full-coalition SSPE is equal to the cNBS.

The sketch of the proof of the first sentence of Corollary 2 is as follows. Suppose that for sufficiently large $p$, for sufficiently large $\delta$, there exists a full-coalition SSPE $s$ of $G(\delta, p)$. Take a sufficiently large $p$. Take a sufficiently large $\delta$. Then, the following argument approximately holds. For any state $\pi$ and any $i \in A^\pi$, let $u^\pi_i$ be player $i$'s expected payoff of $s$ in the subgame with state $\pi$. By her rejection, player $j$'s continuation payoff of her rejection is $(1-p)\,\overline{v}^\pi_{\{i\}} + pu^\pi_i \approx u_i$. In full-coalition SSPE $s$, since player $i$ offers a proposal with the full coalition to be accepted, she obtains a payoff of $\overline{v}^\pi_{A^\pi} - \sum_{j \in A^\pi \setminus \{i\}} \left((1-p)\,\overline{v}^\pi_{\{j\}} + pu^\pi_j\right) \approx \overline{v}^\pi_{A^\pi} - \sum_{j \in A^\pi \setminus \{i\}} u^\pi_j$. If she deviates to offering a proposal with coalition $S$ to be accepted, in full-coalition SSPE $s$, $A^\pi \setminus S$ forms in the next round. Thus, by the deviation, she can obtain $\overline{v}^\pi_S - \sum_{j \in S \setminus \{i\}} u^\pi_j$. Thus, $\overline{v}^\pi_{A^\pi} - \sum_{j \in A^\pi \setminus \{i\}} u^\pi_j \geq \overline{v}^\pi_S - \sum_{j \in S \setminus \{i\}} u^\pi_j$, that is, $\overline{v}^\pi_{A^\pi} - \sum_{j \in A^\pi} u^\pi_j \geq \overline{v}^\pi_S - \sum_{j \in S} u^\pi_j$. Note that since $s$ is a full-coalition SSPE, $\sum_{j \in A^\pi} u^\pi_j = \overline{v}^\pi_{A^\pi}$. Then, $\overline{v}^\pi_S \geq \sum_{j \in S} u^\pi_j$. Note that $S$ is arbitrary and by Theorem 1, the expected payoff profile is the NBS for $(A^\pi, \overline{v}^\pi)$. Thus, the NBS for $(A^\pi, \overline{v}^\pi)$ is in the core of $(A^\pi, \overline{v}^\pi)$. If player $i$ deviates to offering a proposal to be rejected by player $j$, the state is updated to state $\pi \cup \{\{j\}\}$ with probability $1-p$ and it remains $\pi$ with probability $p$. Thus, by the deviation, she can obtain $(1-p)\,u^{\pi \cup \{\{j\}\}}_i + pu^\pi_i$. Thus, $\overline{v}^\pi_{A^\pi} - \sum_{j \in A^\pi \setminus \{i\}} \left((1-p)\,\overline{v}^\pi_{\{j\}} + pu^\pi_j\right) \geq (1-p)\,u^{\pi \cup \{\{j\}\}}_i + pu^\pi_i$, that is, $\overline{v}^\pi_{A^\pi} -$

[7] Suppose that $\pi = \emptyset$. Then, for any $S \in 2^{A^\pi} \setminus \{\emptyset\}$, (3) ((5), resp.) holds if and only if the cNBS (fNBS, resp.) of $(N, V)$ is in the core with merging expectations of $(N, V)$, which is defined by Hafalir (2007).

$p\sum_{j\in A^\pi} u_j^\pi - (1-p)\sum_{j\in A^\pi\setminus\{i\}} \overline{v}_{\{j\}}^\pi \geq (1-p)\, u_i^{\pi\cup\{\{j\}\}}$. Note that since $s$ is a full-coalition SSPE, $\sum_{j\in A^\pi} u_j^\pi = \overline{v}_{A^\pi}^\pi$. Then, $\overline{v}_{A^\pi}^\pi - \sum_{j\in A^\pi\setminus\{i\}} \overline{v}_{\{j\}}^\pi \geq u_i^{\pi\cup\{\{j\}\}}$. Note that by Theorem 1, $u_i^{\pi\cup\{j\}}$ is player $i$'s share in the NBS for $\left(A^\pi\setminus\{j\}, \overline{v}^{\pi\cup\{\{j\}\}}\right)$. Then, (4) holds.

# 6 Delay

Okada (1996), Gomes (2005) and Okada (2010) showed that in their random-proposer coalitional bargaining games, if the grand coalition is efficient, any SSPE involves no delay. Not surprisingly, this result holds in our model if the probability of rejecter-exit partial breakdown is zero, which is formally shown in Proposition 1.

**Proposition 1.** *Let $\delta\in[0,1)$. Suppose that for any subgame $(M,U)$ of $(N,V)$ and any $\pi\in\Pi^M$, $U_{(M,\{M\})}\geq\sum_{S\in\pi}U_{(S,\pi)}$ and the strict inequality holds for $\pi=\{\{j\}\mid j\in M\}$. Then, in any SSPE $s$ of $G(\delta,1)$, any proposal in $s$ is accepted by all responders.*

If the probability of rejecter-exit partial breakdown is positive, even if the bargaining protocol is the random-proposer one and the supposition of Proposition 1 holds, the delay can occur. In the following example, the delay occurs.

**Example 1.** Suppose that $N=\{1,2,3\}$. Suppose that $V_{(\{i\},\{\{j\}|j\in N\})}=0$, $V_{(\{1\},\{\{i\},N\setminus\{1\}\})}=0$, $V_{(\{i\},\{\{i\},N\setminus\{i\}\})}=\frac{1}{2}$ for any $i\in\{2,3\}$, $V_{(S,\{S,N\setminus S\})}=\frac{1}{2}$ for any $S\in 2^N$ with $|S|=2$, and $V_{(N,\{N\})}=1$. Let $\delta\in\left(\frac{4}{5},1\right)$. Let $x$ be the function from $\mathbb{R}^N\times\mathbb{R}$ to $\mathbb{R}^N$ such that for any $(u,p)\in\mathbb{R}^N\times\mathbb{R}$ and $i\in N$,

$$x_i(u,p)=\delta p u_i+\mathbf{1}_{i\in\{2,3\}}\delta(1-p)\frac{1}{2}.$$

Let $g$ be the function from $\mathbb{R}^N\times\mathbb{R}$ to $\mathbb{R}^{\{(i,S)\in N\times 2^N|i\in S\vee S=\emptyset\}}$ such that for any $(u,p)\in\mathbb{R}^N\times\mathbb{R}$, $i\in N$ and $S\in 2^N$ with $S\in i$,[8]

$$g_{(i,S)}(u,p)=\frac{|S|-1}{2}-\sum_{j\in S\setminus\{i\}}x_j(u,p)$$
$$g_{(i,\emptyset)}(u,p)=\delta p u_i+(1-p)\,\delta\frac{1}{4}.$$

[8] In this example, for any $X$, $Y$, $I$, $f:X\to Y^I$, $x\in X$ and $i\in I$, let $f_i(x)$ denote the image of $i$ under "the image of $x$ under $f$", that is, $f_i(x):=(f(x))(i)$.

Let $f$ be the function from $\mathbb{R}^N \times \mathbb{R}$ to $\mathbb{R}^N$ such that for any $(u,p) \in \mathbb{R}^N \times \mathbb{R}$,

$$f_1(u,p) = u_1 - \frac{g_{(1,\emptyset)}(u,p) + 2x_1(u,p)}{3} \tag{6}$$

$$f_2(u,p) = u_2 - \frac{g_{(2,N)}(u,p) + 2x_2(u,p)}{3} \tag{7}$$

$$f_3(u,p) = u_3 - \frac{g_{(3,N)}(u,p) + \left(\delta p u_3 + \delta(1-p)\frac{1}{4}\right) + x_3(u,p)}{3} \tag{8}$$

Let $\bar{u}$ be the element of $\mathbb{R}^N$ such that $\bar{u}_1 = \frac{1}{12}\delta$, $\bar{u}_2 = \frac{1}{3} + \frac{1}{6}\delta$ and $\bar{u}_3 = \frac{1}{3} + \frac{1}{12}\delta$. Then, $f(\bar{u},0) = 0$. The Jacobian matrix of $u \mapsto f(u,p)$ at $(\bar{u},0)$ is the identity matrix and thus is invertible. Therefore, by the implicit function theorem, there exists a neighborhood $D$ of 0 and a continuous function $u$ from $D$ to $\mathbb{R}^N$ such that for any $p \in D$, $f(u(p),p) = 0$. Note that since $\frac{4}{5} < \delta < 1$, $g_{(1,\emptyset)}(u(0),0) - g_{(1,S)}(u(0),0) > 0$ and for any $i \in \{2,3\}$ and $S \in 2^N$ such that $i \in S \neq N$ or $S = \emptyset$, $g_{(i,N)}(u(0),0) - g_{(i,S)}(u(0),0) > 0$. Then, since $g$ is a continuous function, there exists a neighborhood $E$ of 0 with $E \subseteq D$ such that for any $p \in E$, $g_{(1,\emptyset)}(u(p),p) - g_{(1,S)}(u(p),p) > 0$ and for any $i \in \{2,3\}$ and $S \in 2^N$ such that $i \in S \neq N$ or $S = \emptyset$, $g_{(i,N)}(u(p),p) - g_{(i,S)}(u(p),p) > 0$. Let $p \in E$. Let $s$ be the strategy profile in $G(p,\delta)$ defined as follows. In a round with state $\emptyset$, player 1 proposes $(\{1,2\},y)$ with $y_2 = x_2(u(p),p) - 1$; any proposer $i \in \{2,3\}$ proposes $(N,y)$ with $y_j = x_j(u(p),p)$ for any $j \in N \setminus \{i\}$; the last responder $i$ accepts a proposal $(S,y)$ if and only if $y_i \geq x_i(u(p),p)$; another responder accepts a proposal $(S,y)$ if and only if $M = \emptyset$ and $y_i \geq x_i(u(p),p)$, or $M \neq \emptyset$ and $\delta p u_j(p) + \delta(1-p)\frac{1}{4} \geq x_i(u(p),p)$, where $M$ is the set of successors of $j$ who reject the proposal. In a round with state $\pi$ such that $|A^\pi| = 2$, proposer $i$ proposes $(A^\pi, y)$ such that $y_j = \delta p \frac{1}{4}$ for any $j \in A^\pi \setminus \{i\}$; responder $j$ accepts a proposal $(S,x)$ if and only if $y_j \geq \delta p \frac{1}{4}$. Then, player 1's proposal in $s$ is rejected by player 2 in $s$, and the other players' proposals in $s$ are accepted by all responders in $s$. Thus, the expected payoff of $s$ is given by the second terms of right hand sides of equations (6)–(8) at $u = u(p)$. Note that $f(u(p),p) = 0$. Thus, the expected payoff profile is $u(p)$. Hence, the responding actions in $s$ are optimal. Proposer 1's payoff of $s$ is $g_{(1,\emptyset)}(u(p),p)$, and her payoff of any one-stage deviation is less than or equal to $\max\left\{g_{(1,S)} \mid 1 \in S \in 2^N\right\}$. Note that since $p \in E$, $g_{(1,\emptyset)}(u(p),p) > \max\left\{g_{(1,S)} \mid 1 \in S \in 2^N\right\}$. Then, proposer 1's proposal in $s$ is optimal. Let $i \in \{2,3\}$. Proposer $i$'s payoff of $s$ is $g_{(i,N)}(u(p),p)$, and her payoff of any one-stage deviation is less than or equal to $\max\left\{g_{(i,S)} \mid i \in S \in 2^N \setminus \{N\} \vee S = \emptyset\right\}$. Note that since $p \in E$, $g_{(i,N)}(u(p),p) > \max\left\{g_{(i,S)} \mid i \in S \in 2^N \setminus \{N\} \vee S = \emptyset\right\}$. Then, proposer $i$'s proposal in $s$ is optimal. Therefore, $s$ is a subgame perfect equilibrium. $s$ is obviously stationary. Hence, $s$ is an SSPE that involves delay.

The intuitive explanation for the delay is as follows. Under the initial state, since the responders for proposer 1 have strong powers and their acceptances are very "expensive," a proposal to be accepted is not profitable for proposer 1. On the other hand, if she offers a proposal to be rejected by responder 2, responder 2 exits with a high probability, players 1 and 3 have symmetric powers in the next round, and thus, player 1 obtains a moderate amount of expected payoff. Thus, proposer $i$ offers a proposal to be rejected in the first round.

Since there are externalities in Example 1, it is an open question whether the delay occurs when there is the partial breakdown and no externality. It is noteworthy that in the models with other types of partial breakdowns (Miyakawa (2008), Calvo (2008) and Hart and Mas-Colell (1996)), no SSPE involves delay. Since there is no externality in these models, it is not clear whether the difference between the delay in our model and no delay in the their models is due to the type of partial breakdown or the existence of externalities.

# 7 Applications

In this section, we consider mergers of firms in Cournot and Bertrand competitions. Let $N$ be the set of firms. For any $S \in 2^N \setminus \{\emptyset\}$, let $c_S \in \mathbb{R}_+$ be the marginal and average cost for the merged firm of firms in $S$. Suppose that for any $S, T \in 2^N \setminus \{\emptyset\}$, if $S \subseteq T$, then $c_S \geq c_T$. $S \subset T$ and $c_S > c_T$ indicates the synergy effect of reducing cost by merger.

## 7.1 Cournot competition

We consider Cournot competitions. We will demonstrate that there exists a cNBS only if the cost synergy is large but there exists an fNBS. Moreover, we will show that in three-firm case, there exists a full-coalition SSPE if the cost synergy is moderate, and there is less likely to exist a full-coalition SSPE under $\delta \to 1$ than $p = 1$.

Let $P : \mathbb{R}_+ \to \mathbb{R}_+$ be the inverse demand function. For simplicity, suppose that for any $Q \in \mathbb{R}_+$, $P(Q) = \mathbf{1}_{Q \leq 1}(1 - Q)$. For any $\pi \in \Pi^N$, let $\Gamma^{\mathrm{C}}(\pi)$ be the strategic game defined as follows: the set of players is $\pi$; for any $S \in \pi$, the set of player $S$'s strategies is $\mathbb{R}_+$; for any $S \in \pi$, player $S$'s payoff function is $\mathbb{R}_+^\pi \ni q \mapsto \left(P\left(\sum_{S \in \pi} q_S\right) - c_S\right) q_S \in \mathbb{R}$. Then, $G(\pi)$ is a Cournot game played by merged firms. For inner solutions to be ensured, suppose that for any $\pi \in \Pi^N$ and any $S \in \pi$, $\frac{1 + \sum_{T \in \pi} c_T}{|\pi| + 1} > c_S$. For any $\pi \in \Pi^N$, there uniquely exists a Nash equilibrium of $\Gamma^{\mathrm{C}}(\pi)$, and player $S$'s equilibrium payoff is $\left(\frac{1 + \sum_{T \in \pi} c_T}{|\pi| + 1} - c_S\right)^2$. Let $V$ be a map from $\mathcal{C}^N$ to $\mathbb{R}$ such that for any $(S, \pi) \in \mathcal{C}^N$, $V_{(S,\pi)}$ is player $S$'s payoff of the Nash equilibrium in $\Gamma^{\mathrm{C}}(\pi)$.

Since $\sum_{i \in N} V_{(\{i\},\{\{j\}|j \in N\})} \leq V_{(N,\{N\})}$, there uniquely exists an fNBS of $(N, v)$. The fNBS is $x \in \mathbb{R}^N$ such that for any $i \in N$,

$$x_i = \frac{\frac{(1-c_N)^2}{4} - \sum_{j \in N} \left( \frac{1+\sum_{k \in N} c_{\{k\}}}{|N|+1} - c_{\{j\}} \right)^2}{|N|} + \left( \frac{1 + \sum_{k \in N} c_{\{k\}}}{|N| + 1} - c_{\{i\}} \right)^2 .$$

There exists a cNBS if and only if $\frac{(1-c_N)^2}{4} \geq \sum_{i \in N} \frac{\left(1-2c_{\{i\}}+c_{N \setminus \{i\}}\right)^2}{9}$, that is, the profit under monopoly is greater than or equal to the sum of the profits that the firms obtain when they are isolated. If there exists a cNBS, it is $x \in \mathbb{R}^N$ such that for any $i \in N$,

$$x_i = \frac{\frac{(1-c_N)^2}{4} - \sum_{j \in N} \frac{\left(1-2c_{\{j\}}+c_{N \setminus \{j\}}\right)^2}{9}}{|N|} + \frac{\left(1 - 2c_{\{i\}} + c_{N \setminus \{i\}}\right)^2}{9}.$$

Suppose that $|N| \geq 3$. Suppose that for some $c \in \left[0, \frac{4}{2|N|+(-1)^{|N|}+3}\right)$, for any $i \in N$, $c_{\{i\}} = c$ and for any $S \in 2^N \setminus \{\emptyset\}$ with $|S| \geq 2$, $c_S = 0$. Then, there exists a cNBS if and only if $c \geq \frac{2\sqrt{|N|}-3}{4\sqrt{|N|}}$, which means that the cost synergy by merger is significant. Note that $\frac{2\sqrt{|N|}-3}{4\sqrt{|N|}} > 0$, if $|N| \gtreqless 7.5$, $\frac{2\sqrt{|N|}-3}{4\sqrt{|N|}} \gtreqless \frac{4}{2|N|+(-1)^{|N|}+3}$. Then, if there are eight or more firms, regardless of the magnitude of the cost synergy, the cNBS does not exist.

Suppose that $N = \{1, 2, 3\}$. Suppose that for some $c \in \left[0, \frac{1}{2}\right)$, for any $i \in N$, $c_{\{i\}} = c$ and for any $S \in 2^N \setminus \{\emptyset\}$ with $|S| \geq 2$, $c_S = 0$. Then, for some $\bar{p} \in [0, 1)$, for any $p \in [\bar{p}, 1)$, for some $\bar{\delta} \in [0, 1)$, for any $\delta \in \left[\bar{\delta}, 1\right)$, there exists a full-coalition SSPE of $G(\delta, p)$ only if (if, resp.) $\frac{2-\sqrt{3}}{4} \leq c \leq \frac{-2+\sqrt{6}}{2}$ ($\frac{2-\sqrt{3}}{4} < c < \frac{-2+\sqrt{6}}{2}$, resp.). Note that $0 < \frac{2-\sqrt{3}}{4} < \frac{-2+\sqrt{6}}{2} < \frac{1}{2}$. The reasoning is as follows: (i) if $c$ is sufficiently large, a two-firm coalition has a great advantage over the isolated firm, this coalition is profitable, and thus, in state $\pi$ with $A^\pi = N$, the grand coalition fails to be formed; (ii) if $c$ is sufficiently small, by the merger paradox, a two-firm coalition is not profitable, and thus, in state $\pi$ with $|A^\pi| = 2$, the full coalition fails to be formed; (iii) if $c$ is sufficiently small, by the positive externalities, an isolated firm's profit is large, in state $\pi$ with $A^\pi = N$, each responder's continuation payoff is large, and thus, the grand coalition fails to be formed. For some $\bar{\delta} \in [0, 1)$, for any $\delta \in \left[\bar{\delta}, 1\right)$, there exists a full-coalition SSPE of $G(\delta, 1)$ only if (if, resp.) $17-12\sqrt{2} \leq c \leq \frac{-2+\sqrt{6}}{2}$ ($17 - 12\sqrt{2} < c < \frac{-2+\sqrt{6}}{2}$, resp.). Note that $0 < 17 - 12\sqrt{2} < \frac{-2+\sqrt{6}}{2}$. The reasoning behind this condition is the same as (i) and (ii) above. Under $p = 1$, each responder's continuation payoff in state $\pi$ with $A^\pi = N$ is her profit when every player stands alone, the positive externalities do not affect the continuation payoff, and thus, (iii) does not matter. Hence, the condition for a full-coalition SSPE to exist is stronger under $\delta \to 1$ than $p = 1$ (that is, $17 - 12\sqrt{2} < \frac{2-\sqrt{3}}{4}$).

## 7.2 Bertrand competition

We consider Bertrand competitions. We will demonstrate that there exist an cNBS and a fNBS, and if there is a firm that is more efficient than the other firms and there is the cost synergy in merger of the inefficient firms, the most efficient firm's share is smaller in the cNBS than in the fNBS. Moreover, we will show that in three-firm case, there exits a full-coalition SSPE under $\delta \to 1$ only if the cost synergy is small, but there exists always a full-coalition SSPE under $p = 1$,

Let $\epsilon \in \mathbb{R}_{++}$ be the price unit. Let $P = \{\epsilon i \mid i \in \mathbb{Z}_+\}$ be the set of prices. Let $Q : P \to \mathbb{R}_+$ be the demand function. For simplicity, suppose that for any $p \in P$, $Q(p) = \mathbf{1}_{p \leq 1} (1 - p)$. For simplicity, suppose that for any $S \in 2^N \setminus \{\emptyset\}$, $c_S \in P$. For any $\pi \in \Pi^N$, let $\Gamma^{\mathrm{B}}(\pi)$ be the strategic game defined as follows: the set of players is $\pi$; for any $S \in \pi$, the set of player $S$'s strategies is $P$; for any $S \in \pi$, player $S$'s payoff function is $P^\pi \ni p \mapsto \mathbf{1}_{p_S = \min_{T \in \pi} p_T} (p_S - c_S) \frac{Q(p_S)}{|\arg\min_{T \in \pi} p_T|} \in \mathbb{R}$. Then, $\Gamma^{\mathrm{B}}(\pi)$ is a Bertrand game played by merged firms. Suppose that for any $S \in 2^N \setminus \{\emptyset\}$, $c_S < \frac{1}{2}$. By this supposition, in any Nash equilibrium of $\Gamma^{\mathrm{B}}(\pi)$ with $\pi \neq \{N\}$, no player enjoys the monopoly profit. For any $(S, \pi) \in \mathcal{C}^N$, let $c^\pi_{-S} := \min \{c_T \mid T \in \pi \setminus \{S\}\} \cup \left\{\frac{1+c_N}{2}\right\}$. If $S \neq N$, $c^\pi_{-S}$ is the cost of the most efficient competitor for the merged firm of $S$, and if $S = N$, $c^\pi_{-S}$ is the monopoly price. For any $\pi \in \Pi^N$, there exists an undominated Nash equilibrium of $\Gamma^{\mathrm{B}}(\pi)$, and in any such equilibrium, player $S$'s payoff is $\mathbf{1}_{c_S \leq c^\pi_{-S}} \left(c^\pi_{-S} - c_S\right) \left(1 - c^\pi_{-S}\right)$. Let $V$ be a map from $\mathcal{C}^N$ to $\mathbb{R}$ such that for any $(S, \pi) \in \mathcal{C}^N$, $V_{(S,\pi)}$ is player $S$'s payoff of an undominated Nash equilibrium of $\Gamma^{\mathrm{B}}(\pi)$.

Let $i \in \arg\min_{j \in N} c_{\{j\}}$. Since $\sum_{j \in N} V_{(\{j\}, N \setminus \{i\})} \leq \sum_{j \in N} V_{(\{j\}, \{\{k\} | k \in N\})} \leq V_{(N, \{N\})}$, there uniquely exists a cNBS of $(N, v)$. Note that for any $j \in N \setminus \{i\}$, $c_{\{j\}} \geq c_{\{i\}} \geq c_{N \setminus \{j\}}$. The cNBS is $x \in \mathbb{R}^N$ such that

$$x_i = \frac{(1 - c_N)^2 + 4(|N| - 1)\mathbf{1}_{c_{N \setminus \{i\}} > c_{\{i\}}} \left(c_{N \setminus \{i\}} - c_{\{i\}}\right)\left(1 - c_{N \setminus \{i\}}\right)}{4|N|},$$

and for any $j \in N \setminus \{i\}$,

$$x_j = \frac{(1 - c_N)^2 - 4\mathbf{1}_{c_{N \setminus \{i\}} > c_{\{i\}}} \left(c_{N \setminus \{i\}} - c_{\{i\}}\right)\left(1 - c_{N \setminus \{i\}}\right)}{4|N|}.$$

Since $\sum_{j \in N} V_{(\{j\}, \{\{k\} | k \in N\})} \leq V_{(N, \{N\})}$, there uniquely exists an fNBS of $(N, v)$. The fNBS is $x \in \mathbb{R}^N$ such that

$$x_i = \frac{(1 - c_N)^2 + 4(|N| - 1)\left(c^{\{\{j\}|j \in N\}}_{-\{i\}} - c_{\{i\}}\right)\left(1 - c^{\{\{j\}|j \in N\}}_{-\{i\}}\right)}{4|N|},$$

and for any $j \in N \setminus \{i\}$,

$$x_j = \frac{(1 - c_N)^2 - 4\left(c_{-\{i\}}^{\{\{j\}|j\in N\}} - c_{\{i\}}\right)\left(1 - c_{-\{i\}}^{\{\{j\}|j\in N\}}\right)}{4|N|}.$$

Thus, if $c_{N\setminus\{i\}} < c_{-\{i\}}^{\{\{j\}|j\in N\}}$ and $c_{\{i\}} < c_{-\{i\}}^{\{\{j\}|j\in N\}}$, player $i$'s share in the cNBS is smaller than her share in the fNBS; otherwise, the cNBS is equal to the fNBS. Due to the cost synergy, the cost of firm $i$'s (most efficient) competitor in her threat is not greater in the cNBS than in the fNBS, and the other players' payoff in their threats are zero in both the cNBS and the fNBS. Thus, player $i$'s share is not greater in the cNBS than in the fNBS.

Suppose that for any $S, T \in 2^N \setminus \{\emptyset\}$, $c_S = c_T$. Then, for any $S \neq N$, $V_{(S,\pi)} = 0$. Thus, for any $(\delta, p) \in [0, 1) \times [0, 1]$, there exists a full-coalition SSPE of $G(\delta, p)$. Suppose that $N = \{1, 2, 3\}$. Suppose that for some $c \in \left[0, \frac{1}{2}\right)$, $c_{\{1\}} = c_{\{2\}} = c$ and for any $S \neq \{1\}, \{2\}$, $c_S = 0$. Then, for some $\bar{p} \in [0, 1)$, for any $p \in [\bar{p}, 1)$, for some $\bar{\delta} \in [0, 1)$, for any $\delta \in \left[\bar{\delta}, 1\right)$, there exists a full-coalition SSPE of $G(\delta, p)$ only if (if, resp.) $c \leq \frac{3-\sqrt{3}}{6}$ ($c < \frac{3-\sqrt{3}}{6}$, resp.). Note that $0 < \frac{3-\sqrt{3}}{6} < \frac{1}{2}$. If $c$ is large, the advantage of a two-firm coalition that firm 1 belongs to over the other isolated firm is large, this coalition is profitable, and thus, the grand coalition fails to be formed. On the other hand, for some $\bar{\delta} \in [0, 1)$, for any $\delta \in \left[\bar{\delta}, 1\right)$, there exists a full-coalition SSPE of $G(\delta, 1)$. Under $p = 1$, the profit of the two-firm coalition that firm 1 belongs to is large, but since firm 1's threat payoff is large, firm 1's share in the fNBS is large. Thus, the two-firm coalition cannot block the fNBS. Therefore, regardless of $c$, the full-coalition SSPE exists.

# 8 Conclusion

We defined two kinds of NBS of PFGs (fNBS and cNBS). On the basis of any PFG, we defined an EG, which is a propose-respond sequential game where the first rejecter exits from the game with a probability. We showed that in the limit as the discount factor tends to unity (without the partial breakdown, resp.), the cNBS (fNBS, resp.) is supported as the expected payoff profile of any SSPE of the EG such that in any subgame, a coalition of all active players is immediately formed. We also provided a necessary and sufficient condition for such an SSPE to exist. Moreover, we presented an example in which delay of agreements occurs.

# Appendix

# A Lemmas

Let $(\delta, p) \in [0,1) \times [0,1]$. Let $s$ be a full-coalition SSPE of $G(\delta, p)$.

**Lemma 1.** *Let $\pi$ be a state with $A^\pi \neq \emptyset$. For any $i \in A^\pi$, let $u_i$ be player $i$'s expected payoff of $s$ in any subgame starting with state $\pi$. Let $i \in A^\pi$. Let $(S, x)$ be a proposal of player $i$ in a round with state $\pi$. Then, in the round with state $\pi$, if for any $j \in S \setminus \{i\}$, $x_j > (1-\delta)\underline{v}^\pi_{\{j\}} + \delta(1-p)\overline{v}^\pi_{\{j\}} + \delta p u_j$, then, in $s$, player $i$'s proposal $(S, x)$ is accepted by all responders; if for some $j \in S \setminus \{i\}$, $x_j < (1-\delta)\underline{v}^\pi_{\{j\}} + \delta(1-p)\overline{v}^\pi_{\{j\}} + \delta p u_j$, then, in $s$, player $i$'s proposal $(S, x)$ is rejected by some responder.*

*Proof.* Suppose that for any $j \in S \setminus \{i\}$, $x_j > (1-\delta)\underline{v}^\pi_{\{j\}} + \delta(1-p)\overline{v}^\pi_{\{j\}} + \delta p u_j$. Given the other actions in $s$, the last responder $j$ obtains $x_j$ by accepting $(S, x)$ and $(1-\delta) V_{\left(\{j\}, \underline{\pi}^{A^\pi}_{\{j\}} \cup \pi\right)} + \delta(1-p) V_{\left(\{j\}, \overline{\pi}^{A^\pi}_{\{j\}} \cup \pi\right)} + \delta p u_j = (1-\delta)\underline{v}^\pi_{\{j\}} + \delta(1-p)\overline{v}^\pi_{\{j\}} + \delta p u_j > x_j$ by rejecting it. Thus, she accepts it in $s$. Let $j$ be a responder. Suppose that any follower of $j$ accepts $(S, x)$ in $s$. Then, given the other actions in $s$, responder $j$ obtains $x_j$ by accepting $(S, x)$ and $(1-\delta) V_{\left(\{j\}, \underline{\pi}^{A^\pi}_{\{j\}} \cup \pi\right)} + \delta(1-p) V_{\left(\{j\}, \overline{\pi}^{A^\pi}_{\{j\}} \cup \pi\right)} + \delta p u_j = (1-\delta)\underline{v}^\pi_{\{j\}} + \delta(1-p)\overline{v}^\pi_{\{j\}} + \delta p u_j > x_j$ by rejecting it. Thus, she accepts it in $s$. Therefore, by the mathematical induction, $(S, x)$ is accepted by all responders.

Suppose that for some $j \in S \setminus \{i\}$, $x_j < (1-\delta)\underline{v}^\pi_{\{j\}} + \delta(1-p)\overline{v}^\pi_{\{j\}} + \delta p u_j$. Suppose that $(S, x)$ is accepted. Player $j$'s payoff of $s$ at her node at which she responds $(S, x)$ in a round with state $\pi$ is $x_j$. Her payoff of the deviation to rejection is $(1-\delta) V_{\left(\{j\}, \underline{\pi}^{A^\pi}_{\{j\}} \cup \pi\right)} + \delta(1-p) V_{\left(\{j\}, \overline{\pi}^{A^\pi}_{\{j\}} \cup \pi\right)} + \delta p u_j = (1-\delta)\underline{v}^\pi_{\{j\}} + \delta(1-p)\overline{v}^\pi_{\{j\}} + \delta p u_j < x_j$. Thus, the payoff of $s$ is greater than that of the deviation. This is a contradiction. Thus, $(S, x)$ is rejected. Q.E.D.

**Lemma 2.** *Let $\pi$ be a state with $A^\pi \neq \emptyset$. For any $i \in A$, let $u_i$ be player $i$'s expected payoff of $s$ in any subgame starting with state $\pi$ and $\left(S^i, x^i\right)$ be player $i$'s proposal in $s$ in any round with state $\pi$. Then, for any $i \in A^\pi$, $S^i = A^\pi$ and for any $j \in S^i \setminus \{i\}$, $x^i_j = (1-\delta)\underline{v}^\pi_{\{j\}} + (1-p)\delta\overline{v}^\pi_{\{j\}} + p\delta u_j$.*

*Proof.* Since $s$ is full-coalition, $S^i = A^\pi$. Since $s$ is full-coalition, by Lemma 1, for any $j \in S^i \setminus \{i\}$, $x^i_j \geq (1-\delta)\underline{v}^\pi_{\{j\}} + \delta(1-p)\overline{v}^\pi_{\{j\}} + \delta p u_j$. Suppose that for some $j \in S^i$, $x^i_j > (1-\delta)\underline{v}^\pi_{\{j\}} + \delta(1-p)\overline{v}^\pi_{\{j\}} + \delta p u_j$. Let $\epsilon := \frac{x^i_j - (1-\delta)\underline{v}^\pi_{\{j\}} - \delta(1-p)\overline{v}^\pi_{\{j\}} - \delta p u_j}{2}$. Let $y$ be an element in $\mathbb{R}^{S^i}$ such that $y_j = x^i_j - \epsilon$ and for any $k \in S^i \setminus \{j\}$, $y_k = x_k + \frac{\epsilon}{|S^i|-1}$. Then, for any $k \in S^i \setminus \{i\}$, $y_k > (1-\delta)\underline{v}^\pi_{\{k\}} + \delta(1-p)\overline{v}^\pi_{\{k\}} + \delta p u_k$. Thus, by Lemma 1, $\left(y, S^i\right)$ is accepted in $s$. Hence, by the deviation to proposing $\left(y, S^i\right)$, player $i$'s payoff at her proposing node in any round with state $\pi$ increases by $\frac{\epsilon}{|S^i|-1}$, which is a contradiction. Therefore, for any $j \in S^i \setminus \{i\}$, $x^i_j = (1-\delta)\,\underline{v}^\pi_{\{j\}} + (1-p)\,\delta\overline{v}^\pi_{\{j\}} + p\delta u_j$. Q.E.D.

# B Proof of Theorem 1

Let $(\delta, p) \in [0,1) \times [0,1]$. Let $s$ be a full-coalition SSPE of $G(\delta, p)$. Let $\pi$ be a state. For any $i \in A^\pi$, let $u_i$ be player $i$'s expected payoff of $s$ in any subgame starting with state $\pi$. Then, by Lemma 2, for any $i \in A^\pi$,

$$\begin{aligned} u_i = {} & \frac{V_{(A^\pi, \pi\cup\{A^\pi\})} - \sum_{j\in A^\pi\setminus\{i\}} \left((1-\delta)\,\underline{v}^\pi_{\{j\}} + \delta\,(1-p)\,\overline{v}^\pi_{\{j\}} + \delta p u_j\right)}{|A^\pi|} \\ & + (|A^\pi|-1)\,\frac{(1-\delta)\,\underline{v}^\pi_{\{i\}} + \delta\,(1-p)\,\overline{v}^\pi_{\{i\}} + \delta p u_i}{|A^\pi|} \\ = {} & \frac{\overline{v}^\pi_{A^\pi} - \sum_{j\in A^\pi} \left((1-\delta)\,\underline{v}^\pi_{\{j\}} + \delta\,(1-p)\,\overline{v}^\pi_{\{j\}} + \delta p u_j\right)}{|A^\pi|} \\ & + (1-\delta)\,\underline{v}^\pi_{\{i\}} + \delta\,(1-p)\,\overline{v}^\pi_{\{i\}} + \delta p u_i. \end{aligned}$$

and thus,

$$u_i = \frac{\frac{\overline{v}^\pi_{A^\pi} - \delta p \sum_{j\in A^\pi} u_j}{1-\delta p} - \sum_{j\in A^\pi} \frac{(1-\delta)\underline{v}^\pi_{\{j\}} + \delta(1-p)\overline{v}^\pi_{\{j\}}}{1-\delta p}}{|A^\pi|} + \frac{(1-\delta)\,\underline{v}^\pi_{\{i\}} + \delta\,(1-p)\,\overline{v}^\pi_{\{i\}}}{1-\delta p}.$$

Note that since $s$ is a full-coalition SSPE, $\sum_{i\in A^\pi} u_i = V_{(A^\pi,\pi\cup\{A^\pi\})} = \overline{v}^\pi_{A^\pi}$. Then, we have the conclusion. Q.E.D.

# C Proof of Theorem 2

Let $(\delta, p) \in [0,1) \times [0,1]$. For any state $\pi$ with $A^\pi \neq \emptyset$ and any $i \in A^\pi$, let $u^\pi_i := \frac{\overline{v}^\pi_{A^\pi} - \sum_{j\in A^\pi} \frac{(1-\delta)\underline{v}^\pi_{\{j\}} + \delta(1-p)\overline{v}^\pi_{\{j\}}}{1-\delta p}}{|A^\pi|} + \frac{(1-\delta)\underline{v}^\pi_{\{i\}} + \delta(1-p)\overline{v}^\pi_{\{i\}}}{1-\delta p}$ and $x^\pi_i := (1-\delta)\,\underline{v}^\pi_{\{i\}} + \delta\,(1-p)\,\overline{v}^\pi_{\{i\}} + \delta p u^\pi_i$. For any state $\pi$ with $A^\pi \neq \emptyset$, any $i \in A^\pi$ and any $S \in 2^{A^\pi} \setminus \{\emptyset\}$

with $S \ni i$, let

$$
\begin{aligned}
a_{iS}^{\pi} :=& \left( V_{(A^{\pi},\{A^{\pi}\}\cup\pi)} - \sum_{k\in A^{\pi}\setminus\{i\}} x_k^{\pi} \right) \\
&- \left( (1-\delta) V_{\left(S,\underline{\pi}_S^{A^{\pi}}\cup\pi\right)} + \delta V_{\left(S,\overline{\pi}_S^{A^{\pi}}\cup\pi\right)} - \sum_{k\in S\setminus\{i\}} x_k^{\pi} \right) \\
=& \frac{\delta p\,|S| + (1-\delta p)\,|A^{\pi}|}{|A^{\pi}|} \left( \overline{v}_{\{i\}}^{\pi} - \sum_{k\in A^{\pi}} \frac{(1-\delta)\,\underline{v}_{\{k\}}^{\pi} + \delta(1-p)\,\overline{v}_{\{k\}}^{\pi}}{1-\delta p} \right) \\
&- \left( (1-\delta)\,\underline{v}_S^{\pi} + \delta \overline{v}_S^{\pi} - \sum_{k\in S} \frac{(1-\delta)\,\underline{v}_{\{k\}}^{\pi} + \delta(1-p)\,\overline{v}_{\{k\}}^{\pi}}{1-\delta p} \right).
\end{aligned}
$$

For any state $\pi$ with $|A^{\pi}| \geq 2$, any $i \in A^{\pi}$ and any $j \in A^{\pi} \setminus \{j\}$, let

$$
\begin{aligned}
b_{ij}^{\pi} :=& \left( V_{(A^{\pi},\{A^{\pi}\}\cup\pi)} - \sum_{k\in A^{\pi}\setminus\{i\}} x_k^{\pi} \right) \\
&- \left( (1-\delta) V_{\left(\{i\},\underline{\pi}_{\{i\}}^{A^{\pi}}\cup\pi\right)} + \delta(1-p)\, u_i^{\pi\cup\{\{j\}\}} + \delta p u_i^{\pi} \right) \\
=& (1-\delta p) \left( \overline{v}_{A^{\pi}}^{\pi} - \sum_{k\in A^{\pi}} \frac{(1-\delta)\,\underline{v}_{\{k\}}^{\pi} + \delta(1-p)\,\overline{v}_{\{k\}}^{\pi}}{1-\delta p} \right) \\
&- \delta(1-p) \frac{\overline{v}_{A^{\pi}\setminus\{j\}}^{\pi\cup\{\{j\}\}} - \sum_{k\in A^{\pi}\setminus\{j\}} \frac{(1-\delta)\underline{v}_{\{k\}}^{\pi\cup\{\{j\}\}} + \delta(1-p)\overline{v}_{\{k\}}^{\pi\cup\{\{j\}\}}}{1-\delta p}}{|A^{\pi}|-1} \\
&- \delta(1-p) \left( \frac{(1-\delta)\,\underline{v}_{\{i\}}^{\pi\cup\{\{j\}\}} + \delta(1-p)\,\overline{v}_{\{i\}}^{\pi\cup\{\{j\}\}}}{1-\delta p} - v_{\{i\}}^{\pi} \right).
\end{aligned}
$$

**Necessity** Suppose that there exists a full-coalition SSPE $s$ of $G(\delta, p)$. Let $\pi$ be a state with $A^{\pi} \neq \emptyset$. Let $S \in 2^{A^{\pi}} \setminus \{\emptyset\}$. Let $i \in S$. Since $s$ is a full-coalition SSPE, by Lemma 2 and Theorem 1, player $i$'s payoff of $s$ conditional on being a proposer in the round with state $\pi$ is $V_{(A^{\pi},\{A^{\pi}\}\cup\pi)} - \sum_{j\in A^{\pi}\setminus\{i\}} x_j^{\pi}$. For any $\epsilon \in \mathbb{R}_{++}$, let $y^{\epsilon}$ be an element in $\mathbb{R}^S$ such that for any $j \in S \setminus \{i\}$, $y_j^{\epsilon} = x_j^{\pi} + \epsilon$. Then, by Lemma 1 and Theorem 1, for any $\epsilon \in \mathbb{R}_{++}$, player $i$'s proposal $(S, y^{\epsilon})$ is accepted by all responders in $s$. Thus, for any $\epsilon \in \mathbb{R}_{++}$, player $i$'s payoff of the deviation to proposal $(S, y^{\epsilon})$ in the round with state $\pi$ is $(1-\delta) V_{\left(S,\underline{\pi}_S^{A^{\pi}}\cup\pi\right)} + \delta V_{\left(S,\overline{\pi}_S^{A^{\pi}}\cup\pi\right)} - \sum_{j\in S\setminus\{i\}} \left( x_j^{\pi} + \epsilon \right) =: w^{\epsilon}$. Then, for any $\epsilon \in \mathbb{R}_{++}$, since $s$ is a subgame perfect equilibrium, $V_{(A^{\pi},\{A^{\pi}\}\cup\pi)} - \sum_{j\in A^{\pi}\setminus\{i\}} x_j^{\pi} \geq w^{\epsilon}$. Hence, $V_{(A^{\pi},\{A^{\pi}\}\cup\pi)} - \sum_{j\in A^{\pi}\setminus\{i\}} x_j^{\pi} \geq \lim_{\epsilon\to 0} w^{\epsilon}$. Thus, $a_{iS}^{\pi} \geq 0$, which is equivalent to (1). Player $i$'s payoff of the deviation to proposal rejected by player $j$ in $s$ conditional on being a proposer in the round with state $\pi$ is $(1-\delta) V_{\left(\{i\},\underline{\pi}_{\{i\}}^{A^{\pi}}\cup\pi\right)} + \delta(1-p)\, u_i^{\pi\cup\{\{j\}\}} + p u_i^{\pi}$. Then, since $s$ is a subgame perfect equilibrium, $V_{(A^\pi,\{A^\pi\}\cup\pi)}-\sum_{j\in A^\pi\setminus\{i\}} x_j^\pi \geq (1-\delta)\, V_{\left(\{i\},\underline{\pi}_{\{i\}}^{A^\pi}\cup\pi\right)}+\delta\,(1-p)\, u_i^{\pi\cup\{\{j\}\}}+\delta p u_i^\pi$. Thus, $b_{ij}^\pi \geq 0$, which is equivalent to (2).

**Sufficiency** Suppose that for any state $\pi$ with $A^\pi \neq \emptyset$, any $i \in A^\pi$ and any $S \in 2^{A^\pi} \setminus \{\emptyset\}$ with $S \ni i$, (1) holds and for any state $\pi$ with $|A^\pi| \geq 2$, any $i \in A^\pi$ and any $j \in A^\pi \setminus \{i\}$, (2) holds. Then, for any state $\pi$ with $A^\pi \neq \emptyset$, any $i \in A^\pi$ and any $S \in 2^{A^\pi} \setminus \{\emptyset\}$ with $S \ni i$, $a_{iS}^\pi \geq 0$, and for any state $\pi$ with $|A^\pi| \geq 2$, any $i \in A^\pi$ and any $j \in A^\pi \setminus \{j\}$, $b_{ij}^\pi \geq 0$. Construct a strategy profile $s$ of $G(\delta,p)$ as in any round with any state $\pi$, players' actions described as follows. Any proposer $i$ proposes $(A^\pi, x)$ such that for any $j \in A^\pi \setminus \{i\}$, $x_j = x_j^\pi$. Responses to any proposal $(S,x)$ are recursively defined. The last responder $j$ accepts it if and only if $x_j \geq x_j^\pi$. Let $k \in S \setminus \{j\}$. If all followers of $k$ accept it, responder $k$ accepts a proposal $(S,x)$ if and only if $x_k \geq x_k^\pi$; otherwise, she accepts it if and only if $(1-\delta)\,\underline{v}_{\{k\}}^\pi + \delta\,(1-p)\, u_k^{\pi\cup\{\{l\}\}} + \delta p u_k^\pi \geq x_k^\pi$, where $l$ is the first follower who rejects it. Then, any player's proposal in $s$ is accepted in $s$. Consider any player $i$'s proposing node with any state $\pi$. Since her proposal in $s$ is accepted in $s$, her payoff of $s$ at the node is $V_{(A^\pi,\{A^\pi\}\cup\pi)} - \sum_{j\in A^\pi\setminus\{i\}} x_j^\pi$. Her payoff of the deviation to a proposal $(S,x)$ accepted in $s$ is less than or equal to $(1-\delta)\, V_{\left(S,\underline{\pi}_S^{A^\pi}\cup\pi\right)} + \delta V_{\left(S,\overline{\pi}_S^{A^\pi}\cup\pi\right)} - \sum_{j\in S\setminus\{i\}} x_j^\pi$. Since

$$\left(V_{(A^\pi,\{A^\pi\}\cup\pi)} - \sum_{j\in A^\pi\setminus\{i\}} x_j^\pi\right) - \left((1-\delta)\, V_{\left(S,\underline{\pi}_S^{A^\pi}\cup\pi\right)} + \delta V_{\left(S,\overline{\pi}_S^{A^\pi}\cup\pi\right)} - \sum_{j\in S\setminus\{i\}} x_j^\pi\right)$$
$$= a_{iS}^\pi \geq 0,$$

she cannot improve her payoff of this deviation. Player $i$'s payoff of the deviation to a proposal rejected by responder $j$ in $s$ is $(1-\delta)\, V_{\left(\{i\},\underline{\pi}_{\{i\}}^{A^\pi}\cup\pi\right)} + \delta\,(1-p)\, u_i^{\pi\cup\{\{j\}\}} + \delta p u_i^\pi$. Since

$$\left(V_{(A^\pi,\{A^\pi\}\cup\pi)} - \sum_{j\in A^\pi\setminus\{i\}} x_j^\pi\right) - \left((1-\delta)\, V_{\left(\{i\},\underline{\pi}_{\{i\}}^{A^\pi}\cup\pi\right)} + \delta\,(1-p)\, u_i^{\pi\cup\{\{j\}\}} + \delta p u_i^\pi\right)$$
$$= b_{ij}^\pi \geq 0,$$

she cannot improve her payoff of this deviation. Thus, her proposal at the node is optimal. Players' responses in $s$ is obviously optimal. Hence, by the one-stage deviation principle, $s$ is a subgame perfect equilibrium. Obviously, $s$ is a stationary and full-coalition strategy profile. Therefore, $s$ is a full-coalition SSPE. Q.E.D.

# D Proof of Proposition 1

Let $\delta \in [0,1)$. Suppose that for any subgame of $(M,U)$ of $(N,V)$ and any $\pi \in \Pi^M$, $U_{(M,\{M\})} \geq \sum_{S\in\pi} U_{(S,\pi)}$ and the strict inequality holds for $\pi = \{\{i\} \mid i \in M\}$. Let $s$ be an SSPE of $G(\delta,1)$. Let $\pi$ be a state. For any $i \in A^\pi$, let $u_i$ be player $i$'s expected payoff of $s$ in a subgame starting with state $\pi$ and $x_i := (1-\delta)\,\underline{v}_{\{i\}} + \delta u_i$.

**Lemma 3.** *In $s$, any proposal $(S,y)$ such that $y_i > x_i$ for any responder $i$ is accepted by all responders.*

*Proof.* Suppose that in $s$, there exists a responder who rejects $(S,y)$. Let $i$ be the last responder who rejects it. Responder $i$'s payoff of $s$ is $(1-\delta)\,V_{(\{i\},\{\{j\}|j\in A^\pi\}\cup\pi)} + \delta u_i = x_i$. By the deviation to accepting it, she obtains $y_i$. Since $y_i > x_i$, the deviation gain is positive, which is a contradiction. Q.E.D.

**Lemma 4.** $\sum_{i\in A^\pi} u_i \leq V^\pi_{(A^\pi,\{A^\pi\})}$.

*Proof.* Let $h$ be a complete history in a subgame starting with state $\pi$. For any $i \in A^\pi$, let $\bar{u}_i$ be the payoff of player $i$ at $h$. Then, since the transfers are offset, for some $\rho \in \left(\Pi^{A^\pi}\right)^{\mathbb{N}}$, $\sum_{i\in A^\pi} \bar{u}_i = \sum_{t\in\mathbb{N}} (1-\delta)\,\delta^{t-1} \sum_{S\in\rho_t} V^\pi_{(S,\rho_t)}$. By the supposition of the proposition, for any $t \in \mathbb{N}$, $\sum_{S\in\rho_t} V^\pi_{(S,\rho_t)} \leq V_{(A^\pi,\{A^\pi\})}$. Then, $\sum_{i\in A^\pi} \bar{u}_i \leq V^\pi_{(A^\pi,\{A^\pi\})}$. Since complete history $h$ is arbitrary, $\sum_{i\in A^\pi} u_i \leq V^\pi_{(A^\pi,\{A^\pi\})}$. Q.E.D.

By the supposition of the proposition, $\sum_{i\in A^\pi} V^\pi_{(\{i\},\{\{j\}|j\in A^\pi\})} < V_{(A^\pi,\{A^\pi\})}$. Thus, by Lemma 4, $V^\pi_{(A^\pi,\{A^\pi\})} > \sum_{j\in A^\pi} x_j$. Let $\epsilon := \frac{V^\pi_{(A^\pi,\{A^\pi\})} - \sum_{j\in A^\pi} x_j}{2} > 0$. Suppose that there exists a player $i$ such that in $s$, her proposal in $s$ is rejected by some player. Then, her payoff of $s$ at her proposing node is $(1-\delta)\,V_{(\{i\},\{\{j\}|j\in A^\pi\}\cup\pi)} + \delta u_i = x_i$. By the deviation to proposing $(A^\pi, y)$ such that for any $j \in A^\pi \setminus \{i\}$, $y_j = x_j + \frac{\epsilon}{|A^\pi|-1}$, since by Lemma 3, the proposal is accepted by all responders, she obtains payoff $V^\pi_{(A^\pi,\{A^\pi\})} - \sum_{j\in A^\pi\setminus\{i\}} y_j$. By the definitions of $\epsilon$ and $y$, The deviation gain is $\left(V^\pi_{(A^\pi,\{A^\pi\})} - \sum_{j\in A^\pi\setminus\{i\}} y_j\right) - x_i = \epsilon > 0$, which is a contradiction. Q.E.D.

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