Intergenerational Redistribution and the Dynamics of Wealth Distribution*

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Abstract

Using an overlapping-generations model with an age-declining income profile, this paper examines how the intergenerational distribution of consumption, welfare and financial wealth evolves over time. Income redistribution from future generations to the present generation either narrows or widens intergenerational gaps in consumption and welfare depending on the amount of outstanding public bonds. It is also shown that the position of individual financial wealth relative to the social average follows either a humped or monotonically increasing path.

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1 Introduction

Not only aggregate economic performance but also the distribution of income and wealth is the object of increasing interest among policymakers and economists. Recently, Piketty (2014) revives this interest by digging into historical data in various countries. Using an overlapping-generations model with an age-declining income profile, this paper examines how the intergenerational distribution of consumption, welfare and financial wealth dynamically evolves and is affected by public debt issuance and social security systems.

There are numerous theoretical studies that analyze the dynamics of wealth and income distribution. For example, Stiglitz (1969) makes an earlier attempt using the Solow-Swan growth model with nonoptimizing consumers. Ramsey (1928), Becker (1980) and Ikeda and Ono (1992) consider the difference in the rate of time preference to show that the most patient individual eventually owns all financial wealth as a result of rational behavior. Caselli and Ventura (2000) find that various factors other than the rate of time preference —i.e., heterogeneity in individual tastes, skills and initial wealth and macroeconomic factors such as capital accumula-

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tion and technological progress— affect the distribution of consumption, financial wealth and income among rational agents.[1)]

While they focus on the distribution within generations, some studies examine the distributional dynamics across generations using overlapping-generation models. Among them, Diamond (1965) shows that public debt issuance generates the redistribution from future generations to the present generation and deteriorates welfare in the steady state on the dynamically efficient path. Blanchard (1985) and Weil (1989) find that the Ricardian neutrality does not hold even if the present generation infinitely lives because public bond issuance creates persistent income redistribution across generations.[2)] Gertler (1999) incorporates a random transition from work to retirement into the Blanchard model and shows that the redistribution from workers to retirees reduces capital accumulation. Benhabib *et al*. (2014) consider both uninsurable idiosyncratic investment risk and an altruistic bequest motive to analyze wealth distribution.[3)] Simulation results presented by Huggett (1996), Gokhale *et al*. (2001), Heer (2001), De Nardi (2004) emphasize the role of bequests in explaining the wealth inequality in the U.S. economy.[4)]

Bertola *et al*. (2006) assume the hyperbolic absolute risk aversion class of utility functions and investigate the intergenerational distribution of total wealth (i.e., financial wealth plus human wealth) in the presence of capital accumulation and age-declining income profiles. The present paper is closely related to them, but differs in the following three points. First, this paper analytically provides a closed-form solution of the distributional dynamics although the model is a simpler endowment economy where individuals have logarithmic utility. Second, the distribution of not only consumption (or equally, total wealth) but also financial wealth is analyzed. Third, Bertola *et al.* (2006) do not explore policy and welfare implications, whereas this paper examines how intergenerational redistribution policies such as public bond issuance and social security systems affect the distribution and evaluates these effects in terms of individuals' welfare.

The findings are as follows. In the absence of age-declining income profiles, individuals accu-

1) Galor and Zeira (1993) and Matsuyama (2000) analyze the evolution of income classes in the case where there exist imperfect financial markets due to asymmetric information. See Dumas (1989) for heterogeneous risk aversions and Angeletos and Calvet (2006) for incomplete financial markets.

2) Blanchard (1985) takes account of the death probability. However, the models by Blanchard (1985) and Weil (1989) derive basically the same implications because the role of a finite lifetime is negligible, as pointed out by Weil (1987, 1989) and Buiter (1988).

3) The Blanchard–Weil model is employed in various contexts. For example, Ogawa (2005) introduces nominal wage rigidity into the model and shows that debt issuance can be Pareto-improving even on the dynamically-efficient path. Using an endogenous growth model with production externalities, Saint-Paul (1992) finds that public bonds reduce the growth rate and deteriorate the welfare of some future generations. Futagami and Shibata (1998) investigate the sustainability of government budget deficits in a model similar to Saint-Paul. See Weil (1991), Futagami and Shibata (1999, 2000) and Mino and Shibata (2000) for monetary implications in the Blanchard–Weil model.

4) Using overlapping-generations models with altruistic bequest motives, Zilcha (2003), Bossmann *et al*. (2007), Benhabib *et al*. (2011) analytically examine the effects of bequests on the income and wealth inequality.

mulate financial wealth so as to increase their consumption over time. Hence, the positions of individual consumption and financial wealth holdings relative to the social averages dynamically shift from poor classes to wealthy ones. Intergenerational transfers from future generations to the present generation make individuals choose steeper consumption schedules and widens the cross-section gaps in consumption and financial wealth holdings because they stimulate current consumption demand and raise the interest rate. Welfare of future generations decreases and a welfare gap across generations widens.

In the presence of age-declining income profiles, however, the results are fairly different. If the interest rate is sufficiently low due to a small amount of outstanding public bonds, an individual chooses a decreasing path of consumption and a humped path of financial wealth holdings. In this case the relative position of individual consumption to the social average gradually changes from wealthy classes to poor ones, whereas that of individual financial wealth holdings shifts once from poor classes to wealthy ones but eventually returns to poor classes. A rise in the interest rate, which is generated by the intergenerational transfers, induces individuals to choose gentler consumption schedules and narrows the cross-section gaps in consumption and welfare. As the interest rate rises further, the path of individual consumption becomes increasing and the intergenerational redistribution policy in contrast widens the cross-section gaps in consumption and welfare.

The next section constructs an overlapping-generations model with finitely-lived households and derives equilibrium dynamics. Section 3 analyzes the intergenerational distribution of consumption, welfare and financial wealth, and examines the effects of the intergenerational redistribution policy. Section 4 summarizes and concludes this paper.

2 The Structure of the Model

Following Weil (1987, 1989), this paper considers an endowment economy that consists of households with different ages and the government. The government issues public bonds, which generate income redistribution from future generations to the current generation.

2.1 Households

At each point in time a new generation is born at the rate $n(>0)$ and lives forever. The total population at time t is thus given by $N(t)=N(0)e^{nt}$. An individual representative of the generation born at time $s(\leq t)$ maximizes lifetime utility,

$$U(s,\ t)\equiv\int_t^\infty \ln c(s,\ v)e^{-\rho(v-t)}\mathrm{d}v, \tag{1}$$

subject to

$$\frac{\mathrm{d}a(s,\ t)}{\mathrm{d}t}=r(t)a(s,\ t)+y(s,\ t)-c(s,\ t), \tag{2}$$

$$c(s,\ t)\geq 0 \quad \text{for } \forall s\in[0^-,\ t], \tag{3}$$

$$a(s,\ s)=0 \quad \text{for } \forall s\in[0^+,\ t], \tag{4}$$

where $\rho(>0)$, $r(t)$, $c(s,\ t)$, $a(s,\ t)$ and $y(s,\ t)$ respectively denote the subjective discount rate, the interest rate, consumption, financial wealth and noninterest disposable income (i.e., endowments minus lump-sum taxes) at time t of an individual born at time $s(\leq t)$. In the absence of bequest motives, the newly born generation is not linked to preexisting generations and hence, except for the initial generation, financial wealth at birth is zero (see (4)). (Hereafter the initial generation endowed with financial wealth at time 0 is represented by the index 0^- whereas the new generation born at time 0 is indexed by 0^+.)

The optimal conditions are

$$\frac{\mathrm{d}c(s,\ t)}{\mathrm{d}t}=[r(t)-\rho]c(s,\ t), \tag{5}$$

$$\lim_{v\to\infty}\frac{a(s,\ v)}{c(s,\ v)}e^{-\rho(v-t)}=0. \tag{6}$$

The second equation is the transversality condition. Using (2), (5) and (6) yields the individual consumption function,

$$c(s,\ t)=\rho[a(s,\ t)+h(s,\ t)], \tag{7}$$

where $h(s,\ t)$ is individual net human wealth defined as

$$h(s,\ t)\equiv\int_t^\infty y(s,\ v)e^{-\int_t^v r(\mu)\mathrm{d}\mu}\mathrm{d}v. \tag{8}$$

2.2 Aggregation

For any individual variable $x(s,\ t)$, the corresponding per capita aggregate variable $x(t)$ is defined as

$$x(t)\equiv\frac{x(0^-,\ t)N(0^-)+\int_{0^+}^t x(s,\ t)\mathrm{d}N(s)}{N(t)}. \tag{9}$$

Letting $y(t)$ be the social average of noninterest disposable income, $y(s,\ t)$ is assumed to decline with age at the rate $\alpha(>0)$,[5]

$$y(s,\ t)=\begin{cases} y(t)e^{-\alpha t} & \text{for } s=0^-,\\ \dfrac{n+\alpha}{n}y(t)e^{-\alpha(t-s)} & \text{for } s\in[0^+,\ t]. \end{cases} \tag{10}$$

Blanchard (1985) interprets this age-declining schedule as an approximation of retirement, whereas Saint-Paul (1992) considers that the introduction of social security systems decreases α since it makes the income profile shift toward old ages. Note that $\alpha=0$ is a unique consistent schedule when $n=0$.

Substitute (10) into (8) to show that individual human wealth also becomes agedeclining:

$$h(s,\ t)=\begin{cases} e^{-\alpha t}\displaystyle\int_t^\infty y(v)e^{-\int_t^v[r(\mu)+\alpha]\mathrm{d}\mu}\mathrm{d}v & \text{for } s=0^-,\\ e^{-\alpha(t-s)}\dfrac{n+\alpha}{n}\displaystyle\int_t^\infty y(v)e^{-\int_t^v[r(\mu)+\alpha]\mathrm{d}\mu}\mathrm{d}v & \text{for } s\in[0^+,\ t]. \end{cases} \tag{11}$$

Applying the aggregation rule (9) to (11) gives the per capita aggregate human wealth:

$$h(t)=\int_t^\infty y(v)e^{-\int_t^v[r(\mu)+\alpha]\mathrm{d}\mu}\mathrm{d}v, \tag{12}$$

and hence

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5) The consistency between the individual schedule, $y(s,\ t)$, and the per capita aggregate one, $y(t)$, is confirmed by substituting (10) into (9).

$$\frac{\mathrm{d}h(t)}{\mathrm{d}t}=[r(t)+\alpha]h(t)-y(t). \tag{13}$$

Note that age-declining individual human wealth (11) means a rise in the discount rate on future noninterest disposable income in terms of the per capita aggregate value.

Using (9), the aggregations of (7) and (2) respectively yield

$$c(t)=\rho[a(t)+h(t)], \tag{14}$$

$$\frac{\mathrm{d}a(t)}{\mathrm{d}t}=[r(t)-n]a(t)+y(t)-c(t). \tag{15}$$

Differentiating (14) with respect to time and substituting (13) and (15) into the result gives the dynamic equation of per capita aggregate consumption:

$$\frac{\mathrm{d}c(t)}{\mathrm{d}t}=[r(t)+\alpha-\rho]c(t)-(n+\alpha)\rho a(t). \tag{16}$$

2.3 The government and market equilibrium

The government keeps public debt at constant level $\bar{b}(\geq 0)$ and imposes lump-sum tax $\tau(t)$, which is measured in the aggregate per capita term, to finance interest payments:

$$\tau(t)=[r(t)-n]\bar{b}. \tag{17}$$

Since public debt is a unique financial wealth in this economy, the equilibrium condition in the financial market is given by

$$a(t)=\bar{b}. \tag{18}$$

Assuming that the economy receives constant endowment $\bar{e}(>0)$ in the aggregate per capita term, the equilibrium condition in the commodity market satisfies

$$c(t)=\bar{e} \qquad (y(t)=\bar{e}-\tau(t)). \tag{19}$$

Substituting (18) and (19) into (16) implies that there is no transition of aggregate dynamics and determines the equilibrium interest rate r^* as

$$\begin{aligned} r^* &= \frac{(n+\alpha)\rho}{\bar{e}}\bar{b}+(\rho-\alpha) \\ &= -\frac{\rho}{\bar{e}}\left(\frac{\bar{e}}{\rho}-\bar{b}\right)\alpha+\frac{(\bar{e}+n\bar{b})\rho}{\bar{e}}. \end{aligned} \tag{20}$$

It must satisfy $\rho-\alpha\leq r^*\leq\rho+n$, where the first inequality comes from (20) in which $\bar{b}=0$ while the second inequality is the condition for keeping individual consumption nonnegative ($c(s, t)\geq 0$).[6] To obtain economically meaningful results ($c(s, t)>0$), this paper imposes a more restricted condition,

$$\rho-\alpha\leq r^*<\rho+n, \quad \text{or equivalently,}\ 0\leq\bar{b}<\frac{\bar{e}}{\rho}. \tag{21}$$

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6) From (4), (5) and (7), $c(s, t)$ is nonnegative if and only if the initial individual human capital $h(s, s)$ is nonnegative for $\forall s\in[0^+, \infty)$. Furthermore, (11) and (12) shows that the sign of $h(s, s)$ corresponds to that of $h(s)$. Substituting (18) and (19) into (14) implies that $h(s)\geq 0$ holds if $\frac{\bar{e}}{\rho}\geq\bar{b}$, which gives the upper bound of public bond issuance. Using (20) to eliminate $\bar{b}$ from this condition obtains $\rho+n\geq r^*$. See also Blanchard (1985) and Weil (1987, 1989) for this condition.

From (20) and (21), the following lemma is established:

Lemma 1. *Public debt issuance (i.e., an increase in* $\bar{b}$*) and introducing social security systems (i.e., a decrease in* α*) raise the equilibrium interest rate,* r^*.

The intuition behind this lemma is as follows. Public debt issuance creates income transfers from future generations to the present generation, which increase current consumption demand and reduce saving motives. Then, the equilibrium interest rate rises. Similarly introducing social security systems mitigates the age-declining income schedule and decreases saving motives, therefore leading to a rise in the equilibrium interest rate.[7]

3 Distribution

This section analyzes the dynamics of the intergenerational distribution of consumption, welfare and financial wealth. Following Stiglitz (1969) and Caselli and Ventura (2000), I define an individual variable relative to the social average as

$$x^R(s,\ t) \equiv \frac{x(s,\ t)}{x(t)}, \tag{22}$$

which gives the cross-section distribution among generations at time t,

$$x^R(0,\ t),\ \cdots,\ x^R(t,\ t)).$$

In the present paper x takes either consumption c or financial wealth a. Individuals are said to be relatively wealthy (poor) if $x^R(s,\ t) > (<) 1$.

3.1 Distribution of consumption

Let me first analyze the dynamic behavior of relative consumption, $c^R(s,\ t) \equiv \dfrac{c(s,\ t)}{c(t)}$. Because (19) holds, the dynamics of $c^R(s,\ t)$ follow those of $c(s,\ t)$, (5), in the present setting. Taking account of (21), they are

$$\frac{\mathrm{d}c^R(s,\ t)}{\mathrm{d}t} \Big/ c^R(s,\ t) = r^* - \rho \begin{cases} <0 & \text{if } \rho-\alpha \le r^* < \rho, \\ \ge 0 & \text{if } \rho \le r^* < \rho+n, \end{cases} \tag{23}$$

which, as formally shown in appendix 1, has the following properties: for $\forall s \in [0^+,\ \infty)$,

$$c^R(0^-,\ 0^-) = 1, \tag{24}$$

$$c^R(s,\ s) = 1 - \frac{r^* - \rho}{n} \begin{cases} >1 & \text{if } \rho-\alpha \le r^* < \rho, \\ \le 1 & \text{if } \rho \le r^* < \rho+n. \end{cases} \tag{25}$$

In the overlapping-generations model in which new generations continuously emerge, the dynamic behavior of individual consumption differs from that of the social average, which is constant over time in the present setting. Hence, the relative position of individual consumption $c^R(s,\ t)$ changes even in the steady state. Figure 1 and 2 describe the dynamic path of $c^R(s,\ s)$ for $\forall s \in [0^+,\ \infty)$. When the interest rate is low enough to satisfy $r^* < \rho$, an individual chooses a declining consumption path and hence the relative position gradually shifts from

7) Note that if $n=0$ (and $\alpha=0$), the Ricardian neutrality holds and the interest rate remains constant at the level characterized by the modified golden rule (i.e., $r^*=\rho$).

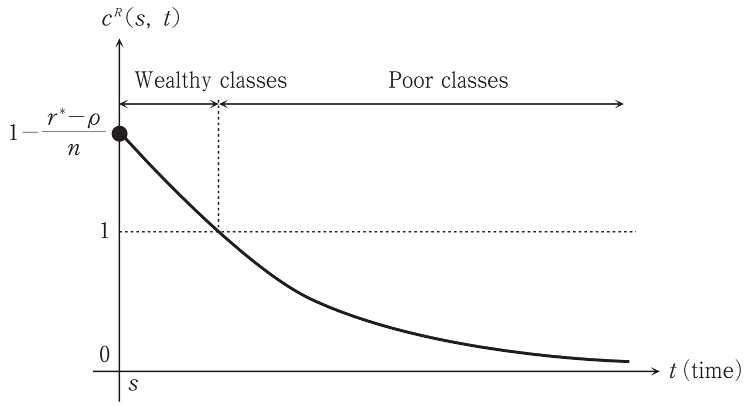

Figure 1: The dynamics of relative consumption for $\forall s \in [0^+,\ \infty)$ if $\rho-\alpha \leq r^* < \rho$.

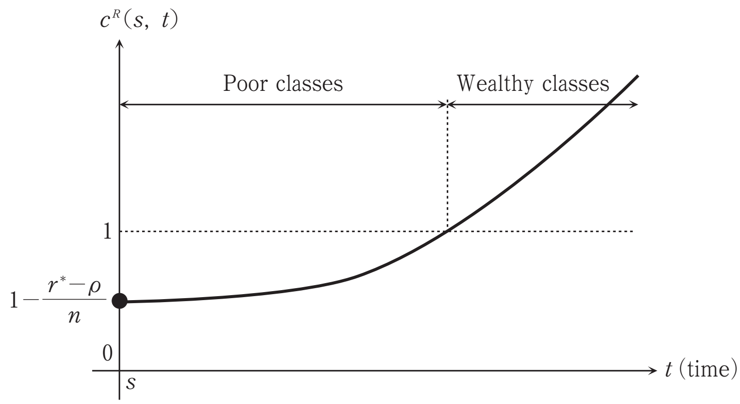

Figure 2: The dynamics of relative consumption for $\forall s \in [0^+,\ \infty)$ if $\rho < r^* < \rho+n$.

wealthy consumption classes to poor ones (see figure 1). As the interest rate rises within $r^* \leq \rho$ due to either public debt issuance or social security systems, the individual consumption profile becomes flat and the cross-section distribution of individual consumption moves toward equality. As the interest rate rises up to $r^* > \rho$, however, the consumption profile becomes increasing. The relative position starts from poor classes and moves to wealthy ones over lifetime (see figure 2). In contrast with the case where $r^* < \rho$, a further rise in the interest rate makes the cross-section distribution of consumption go to inequality.[8)]

Taking lemma 1 into account, I can summarize the result in the following proposition:

Proposition 1. *If the amount of outstanding public bonds is low (resp. high) enough to satisfy* $0 \leq \bar{b} < \dfrac{\alpha \bar{e}}{(n+\alpha)\rho}$ $\left(\textit{resp.}\ \dfrac{\alpha \bar{e}}{(n+\alpha)\rho} \leq \bar{b} < \dfrac{\bar{e}}{\rho}\right)$ *or equivalently* $\rho-\alpha \leq r^* < \rho$ (*resp.* $\rho \leq r^* <$

8) Note that $c^R(0^-,\ 0^-)$ starts from 1 and is monotonically decreasing or increasing depending on the sign of $r^*-\rho$.

$\rho+n$), public debt issuance and introducing social security systems narrow (resp. widen) the cross-section consumption gap across generations.

Remark 1. *If $y(s, t)$ is flat over lifetime (i.e., $\alpha=0$), the intergenerational redistribution policy definitely widens the intergenerational gap in consumption.*

3.2 Distribution of welfare

Let me next examine the distribution of welfare. From (1) and (5), I have

$$U(s, s)=\frac{\ln c(s, s)}{\rho}+\frac{r^*-\rho}{\rho^2}.$$

Keeping (19) and (22) in which $x=c$ in mind, (24) and (25) reduces the welfare to

$$U(s, s)=\begin{cases}\dfrac{\ln\bar{e}}{\rho}+\dfrac{r^*-\rho}{\rho^2} & \text{for } s=0^-,\\[2ex] \dfrac{\ln\bar{e}}{\rho}+\dfrac{r^*-\rho}{\rho^2}+\dfrac{1}{\rho}\ln\left(1-\dfrac{r^*-\rho}{n}\right) & \text{for } \forall s\in[0^+, \infty),\end{cases} \tag{26}$$

which gives a welfare gap between the present generation and future generations:

$$U(0^-, 0^-)-U(s, s)=-\frac{1}{\rho}\ln\left(1-\frac{r^*-\rho}{n}\right)\begin{cases}<0 & \text{if } \rho-\alpha\leq r^*<\rho,\\ \geq 0 & \text{if } \rho\leq r^*<\rho+n,\end{cases}$$

where $s\in[0^+, \infty)$. Under the condition (21), it is increasing with respect to r^*:

$$\frac{\partial[U(0^-, 0^-)-U(s, s)]}{\partial r^*}=\frac{1}{(\rho+n-r^*)\rho}>0.$$

The initial generation is endowed with nonnegative financial wealth at birth, whereas future generations receive higher noninterest disposable income when young in the presence of age-declining income profiles. The relative magnitude of welfare between the initial generation and future generations depends on the level of the interest rate and varies in response to policy changes.

Therefore, together with lemma 1, I obtain the following proposition:

Proposition 2. *If the amount of outstanding public bonds is low (resp. high) enough to satisfy $0\leq\bar{b}<\dfrac{\alpha\bar{e}}{(n+\alpha)\rho}\left(\text{resp. } \dfrac{\alpha\bar{e}}{(n+\alpha)\rho}\leq\bar{b}<\dfrac{\bar{e}}{\rho}\right)$ or equivalently $\rho-\alpha\leq r^*<\rho$ (resp. $\rho\leq r^*<\rho+n$), the welfare of future generations is larger (resp. smaller) than that of the present generation, and public debt issuance and introducing social security systems narrow (resp. widen) the inter-generational gap in welfare.*

Remark 2. *If $y(s, t)$ is age-independent (i.e., $\alpha=0$), the intergenerational redistribution policy definitely increases the intergenerational gaps in both consumption and welfare.*

Remark 3. *Propositions 1 and 2 implies that the intergenerational redistribution policy, which is designed to narrow the cross-section consumption gap, simultaneously attains the intergenerational equalization of welfare.*

The redistribution policy may be beneficial or harmful to future generations because (26) satisfies

$$\frac{\partial U(0^-,\ 0^-)}{\partial r^*}=\frac{1}{\rho^2}>0,$$

$$\frac{\partial U(s,\ s)}{\partial r^*}=-\frac{r^*-n}{(\rho+n-r^*)\rho^2}\begin{cases}>0 & \text{if } \rho-\alpha\leq r^*<n,\\ \leq 0 & \text{if } \max\{\rho-\alpha,\ n\}\leq r^*<\rho+n,\end{cases}$$

where $s\in[0^+,\ \infty)$. As is well known, intergenerational transfers from future generations to the present generation remedy oversavings and are Pareto-improving in a Ponzi scheme ($r^*<n$), whereas they deteriorate the welfare of future generations in a No-Ponzi scheme ($r^*\geq n$).[9] In the case where $\rho-\alpha\leq r^*<\min\{\rho,\ n\}$, the intergenerational redistribution policy is Pareto-improving and simultaneously narrows the welfare gap across generations. In contrast, if $\max\{\rho,\ n\}\leq r^*<\rho+n$, it deteriorates the welfare of future generations widening the welfare gap.

3.3 Distribution of financial wealth

In the real world policymakers and economists are concerned about the distribution of not only consumption but also financial wealth. It is impossible to define $a^R(s,\ t)$ in the case of $a(t)=\bar{b}=0$, so that this subsection imposes the following condition in place of (21):

$$\rho-\alpha<r^*<\rho+n,\quad \text{or equivalently,}\quad 0<\bar{b}<\frac{\bar{e}}{\rho}. \tag{27}$$

Since $\dfrac{\mathrm{d}a^R(s,\ t)}{\mathrm{d}t}=\dfrac{\mathrm{d}a(s,\ t)}{\mathrm{d}t}\Big/\bar{b}$ holds from (18) and (22) in which $x=a$, the time derivative of (7) gives

$$\frac{\mathrm{d}a^R(s,\ t)}{\mathrm{d}t}=\frac{c(s,\ t)}{\rho\bar{b}}\left(\frac{\mathrm{d}c(s,\ t)}{\mathrm{d}t}\Big/c(s,\ t)\right)-\frac{h(s,\ t)}{\bar{b}}\left(\frac{\mathrm{d}h(s,\ t)}{\mathrm{d}t}\Big/h(s,\ t)\right).$$

From (14), (18) and (19), $h(t)$ is constant over time—i.e.,

$$h(t)=\frac{\bar{e}}{\rho}-\bar{b}, \tag{28}$$

and thus the time derivative of (11) implies

$$\frac{\mathrm{d}h(s,\ t)}{\mathrm{d}t}\Big/h(s,\ t)=-\alpha.$$

The use of this equation and (5) reduces the dynamic equation of $a^R(s,\ t)$ to

$$\frac{\mathrm{d}a^R(s,\ t)}{\mathrm{d}t}=\frac{c(s,\ t)}{\rho\bar{b}}\left[(r^*-\rho)+\alpha\frac{\rho h(s,\ t)}{c(s,\ t)}\right]\begin{cases}\gtreqless 0 & \text{if } \rho-\alpha<r^*<\rho,\\ >0 & \text{if } \rho\leq r^*<\rho+n.\end{cases} \tag{29}$$

In the case of $r^*>\rho$, an individual chooses to increase consumption over lifetime facing age-declining human wealth, so that they accumulate financial wealth. As a result, $a^R(s,\ t)$ monotonically shifts from poor classes to wealthy ones (see figure 3). By contrast, in the case of $r^*<\rho$ the sign of (29) is ambiguous because paths of consumption and human wealth are both decreasing. Appendix 2 derives a closed-form solution of (29) and finds that $a^R(s,\ t)$ takes a humped path—i.e., $a^R(s,\ t)$ initially rises but eventually falls (see figure 4).

9) See Diamond (1965) and Weil (1989).

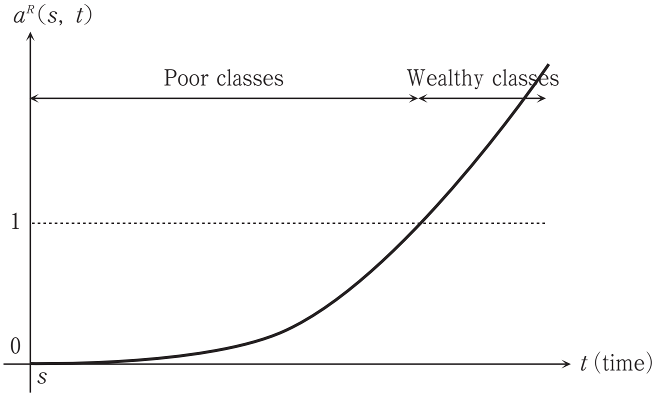

Figure 3: The dynamics of relative financial wealth for $\forall s \in [0^+,\ \infty)$ if $\rho \leq r^* < \rho + n$ and $\dfrac{d^2 a^R(s,\ t)}{dt^2} > 0$ for $\forall t \in [s,\ \infty)$.

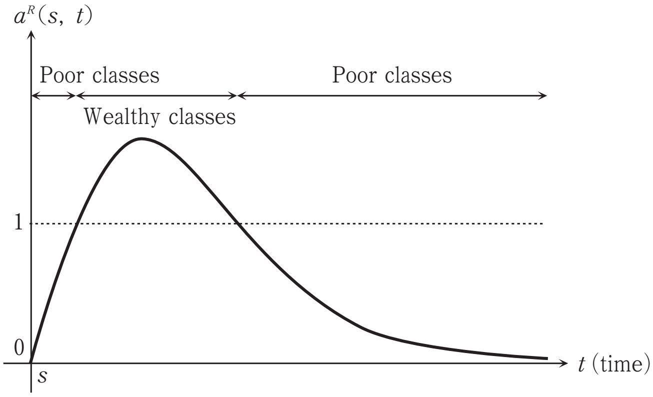

Figure 4: The dynamics of relative financial wealth for $\forall s \in [0^+,\ \infty)$ if $\rho - \alpha < r^* < \rho$.

Taking account of lemma 1, I can summarize the result in the following proposition:

Proposition 3. *The relative position of individual financial wealth follows a humped path if the amount of outstanding public bonds is low enough to satisfy* $0 < \bar{b} < \dfrac{\alpha \bar{e}}{(n+\alpha)\rho}$ *or equivalently* $\rho - \alpha < r^* < \rho$*; whereas it is monotonically increasing if* $\dfrac{\alpha \bar{e}}{(n+\alpha)\rho} \leq \bar{b} < \dfrac{\bar{e}}{\rho}$ *or equivalently* $\rho \leq r^* < \rho + n$*. Public debt issuance and introducing social security systems change the dynamics from the former to the latter.*

Remark 4. *If* $y(s,\ t)$ *is age-independent (i.e.,* $\alpha = 0$*), the relative position of individual financial wealth, as well as that of individual consumption, definitely takes a monotonically increasing* *path.*

Remark 5. *Propositions 1-3 implies that the intergenerational redistribution policy, which achieves the equalizations of consumption and welfare across generations, does not simultaneously lead to the intergenerational equalization of financial wealth. This is because* $c^R(s,\ t)=1$ *and* $U(0^-,\ 0^-)=U(s,\ s)$ *but* $a^R(s,\ t)>1$ *in the case of* $r^*=\rho$.

4 Conclusion

Using an overlapping-generations model with infinitely-lived households, this paper shows that the intergenerational redistribution policy such as public debt issuance and social security systems may narrow or widen the intergenerational gaps in consumption, welfare and financial wealth depending on the amount of outstanding public bonds. If the interest rate is sufficiently low due to a small amount of outstanding bonds, a rise in the interest rate, which is caused by the redistribution policy, induces individuals to choose flatter consumption schedules and narrows the intergenerational gaps in consumption and welfare. As public bonds accumulate and the interest rate rises further, the redistribution policy in contrast widens the intergenerational gaps making individuals choose increasing and steeper consumption schedules. This paper also finds that, depending on the amount of outstanding bonds, the dynamics of relative financial wealth have either a humped path or monotonically increasing path. The intergenerational redistribution policy, which achieves the equalizations of consumption and welfare across generations, does not simultaneously lead to the intergenerational equalization of financial wealth.

There are various directions of extensions. Only three are mentioned here. First, considering capital accumulation, which yields transitional dynamics of aggregate variables, is useful to analyze the relation between economic growth and wealth distribution. Second, introducing heterogeneity in endowments and tastes enables us to examine the *intragenerational* distribution (see e.g., Caselli and Ventura 2000). Finally, imperfect and incomplete financial markets, which lead to inefficient income and wealth allocations, may bring different policy implications.

Appendices

Appendix 1. The derivation of (24) and (25)

This appendix derives (24) and (25) in the text. Equation (24) is obvious because $c(0^-,\ 0^-)=c(0^-)$ holds from (9) and $c(0^-)$ is given by (19).

Substituting (4) into (7) in which $t=s$ and using the result to eliminate $c(s,\ s)$ from (22) in which $x=c$ yields

$$c^R(s,\ s)=\frac{\rho h(s,\ s)}{c(s)}. \tag{A. 1}$$

Substituting (12) into the second equation in (11) in which $t=s$ and rearranging (14) respectively provide

$$h(s,\ s)=\frac{n+\alpha}{n}h(s),\quad h(s)=\frac{c(s)}{\rho}-a(s).$$

Since in the steady state (16) satisfies

$$a(s)=\frac{(r^{*}+\alpha-\rho)c(s)}{(n+\alpha)\rho},$$

successive substitution of the above three equations into (A. 1) obtains (25); i.e.,

$$c^{R}(s,\ s)=\frac{\rho+n-r^{*}}{n}=1-\frac{r^{*}-\rho}{n}.$$

Appendix 2. The dynamics of relative financial wealth

This appendix gives a closed-form solution of (29). Substituting (12) into (11) and applying (28) to the result provides

$$h(s,\ t)=\begin{cases}\left(\dfrac{\bar{e}}{\rho}-\bar{b}\right)e^{-\alpha t} & \text{for } s=0^{-},\\ \dfrac{n+\alpha}{n}\left(\dfrac{\bar{e}}{\rho}-\bar{b}\right)e^{-\alpha(t-s)} & \text{for } \forall s\in[0^{+},\ \infty).\end{cases} \tag{A. 2}$$

Integrate (5) from period s to period t to obtain

$$c(s,\ t)=c(s,\ s)e^{(r^{*}-\rho)t},$$

where $c(s,\ s)$ is given by substituting $a(0^{-},\ 0^{-})=\bar{b}$, (4) and (A. 2) in which $t=s$ into (7) in which $t=s$:

$$c(s,\ s)=\begin{cases}\bar{e} & \text{for } s=0^{-},\\ \dfrac{(n+\alpha)\rho}{n}\left(\dfrac{\bar{e}}{\rho}-\bar{b}\right) & \text{for } \forall s\in[0^{+},\ \infty).\end{cases}$$

The use of these three equations to eliminate $c(s,\ t)$ and $h(s,\ t)$ rewrites (29) as

$$\frac{\mathrm{d}a^{R}(s,\ t)}{\mathrm{d}t}=\begin{cases}\dfrac{\bar{e}}{\rho\bar{b}}(r^{*}-\rho)e^{(r^{*}-\rho)t}+\dfrac{\alpha}{\bar{b}}\left(\dfrac{\bar{e}}{\rho}-\bar{b}\right)e^{-\alpha t} & \text{for } s=0^{-},\\ \dfrac{n+\alpha}{n\bar{b}}\left(\dfrac{\bar{e}}{\rho}-\bar{b}\right)[(r^{*}-\rho)e^{(r^{*}-\rho)(t-s)}+\alpha e^{-\alpha(t-s)}] & \text{for } \forall s\in[0^{+},\ \infty),\end{cases} \tag{A. 3}$$

which provides

$$a^{R}(s,\ t)=\begin{cases}\dfrac{\bar{e}}{\rho\bar{b}}e^{(r^{*}-\rho)t}-\dfrac{1}{\bar{b}}\left(\dfrac{\bar{e}}{\rho}-\bar{b}\right)e^{-\alpha t} & \text{for } s=0^{-},\\ \dfrac{n+\alpha}{n\bar{b}}\left(\dfrac{\bar{e}}{\rho}-\bar{b}\right)[e^{(r^{*}-\rho)(t-s)}-e^{-\alpha(t-s)}] & \text{for } \forall s\in[0^{+},\ \infty),\end{cases} \tag{A. 4}$$

$$\frac{\mathrm{d}^{2}a^{R}(s,\ t)}{\mathrm{d}t^{2}}=\begin{cases}\dfrac{\bar{e}}{\rho\bar{b}}(r^{*}-\rho)^{2}e^{(r^{*}-\rho)t}-\dfrac{\alpha^{2}}{\bar{b}}\left(\dfrac{\bar{e}}{\rho}-\bar{b}\right)e^{-\alpha t} & \text{for } s=0^{-},\\ \dfrac{n+\alpha}{n\bar{b}}\left(\dfrac{\bar{e}}{\rho}-\bar{b}\right)[(r^{*}-\rho)^{2}e^{(r^{*}-\rho)(t-s)}-\alpha^{2}e^{-\alpha(t-s)}] & \text{for } \forall s\in[0^{+},\ \infty).\end{cases} \tag{A. 5}$$

Together with (20) and (27), these equations characterize the dynamic behavior of $a^{R}(s,\ t)$, which varies according to the following three cases:

(i) $\rho<r^{*}<\rho+n$, or equivalently, $\dfrac{\alpha\bar{e}}{(n+\alpha)\rho}<\bar{b}<\dfrac{\bar{e}}{\rho}$.

(ii) $r^{*}=\rho$, or equivalently, $\bar{b}=\dfrac{\alpha\bar{e}}{(n+\alpha)\rho}$.

(iii) $\rho-\alpha<r^{*}<\rho$, or equivalently, $0<\bar{b}<\dfrac{\alpha\bar{e}}{(n+\alpha)\rho}$.

Case (i): Begin with the case (i). Since $r^{*}>\rho$ and $\bar{b}<\dfrac{\bar{e}}{\rho}$ in this case, (A. 3) and (A. 4) respectively satisfy

$$\frac{\mathrm{d}a^R(s,\ t)}{\mathrm{d}t}>0 \qquad \text{for } \forall s\in[0^-,\ \infty),$$

$$a^R(s,\ s)=\begin{cases}1 & \text{for } s=0^-,\\ 0 & \text{for } \forall s\in[0^-,\ \infty),\end{cases}$$

$$\lim_{t\to\infty} a^R(s,\ t)=\infty \quad \text{for } \forall s\in[0^-,\ \infty).$$

Therefore, the relative financial wealth of all generations is monotonically increasing and diverges to infinity. Note that the sign of (A. 5) is ambiguous in this case. Figure 3 depicts the dynamic behavior of $a^R(s,\ t)$ for $\forall s\in[0^+,\ \infty)$ when $\frac{\mathrm{d}^2a^R(s,\ t)}{\mathrm{d}t^2}$ has a positive sign—i.e., for future generations, the relative position of financial wealth holdings starts from poor classes and ends up wealthy classes.

Case (ii): Turn to the case (ii) where $r^*=\rho$ and $\bar{b}=\frac{\alpha\bar{e}}{(n+\alpha)\rho}$. In this case the signs of (A. 3)–(A. 5) are determined:

$$\frac{\mathrm{d}a^R(s,\ t)}{\mathrm{d}t}>0 \qquad \text{for } \forall s\in[0^-,\ \infty),$$

$$a^R(s,\ s)=\begin{cases}1 & \text{for } s=0^-,\\ 0 & \text{for } \forall s\in[0^+,\ \infty),\end{cases}$$

$$\lim_{t\to\infty} a^R(s,\ t)=1+\frac{n}{\alpha}(>1) \quad \text{for } \forall s\in[0^-,\ \infty),$$

$$\frac{\mathrm{d}^2a^R(s,\ t)}{\mathrm{d}t^2}<0 \qquad \text{for } \forall s\in[0^-,\ \infty).$$

For any generation, the relative financial wealth is monotonically increasing and asymptotically approaches $1+\frac{n}{\alpha}$.

Case (iii): The case (iii) is that $r^*<\rho$ and $\bar{b}<\frac{\alpha\bar{e}}{(n+\alpha)\rho}<\frac{\bar{e}}{\rho}$. (A. 4) satisfies

$$a^R(s,\ s)=\begin{cases}1 & \text{for } s=0^-,\\ 0 & \text{for } \forall s\in[0^-,\ \infty),\end{cases}$$

$$\lim_{t\to\infty} a^R(s,\ t)=\infty \quad \text{for } \forall s\in[0^-,\ \infty).$$

Hence, the relative financial wealth of any generation ends up poor classes.

Substituting (20) into (A. 3) and (A. 5) for $s=0^-$ yields

$$\frac{\mathrm{d}a^R(0^-,\ t)}{\mathrm{d}t}=\frac{\alpha e^{-\alpha t}}{\bar{b}}\left\{\left(\frac{\bar{e}}{\rho}-\bar{b}\right)-\underbrace{\frac{n+\alpha}{\alpha}\left[\frac{\alpha\bar{e}}{(n+\alpha)\rho}-\bar{b}\right]}_{(+)}e^{\frac{(n+\alpha)\rho\bar{b}}{\bar{e}}t}\right\},$$

$$\frac{\mathrm{d}^2a^R(0^-,\ t)}{\mathrm{d}t^2}=\frac{\alpha^2e^{-\alpha t}}{\bar{b}}\left\{\underbrace{\frac{(n+\alpha)^2\rho}{\alpha^2\bar{e}}\left[\frac{\alpha\bar{e}}{(n+\alpha)\rho}-\bar{b}\right]^2}_{(+)}e^{\frac{(n+\alpha)\rho\bar{b}}{\bar{e}}t}-\left(\frac{\bar{e}}{\rho}-\bar{b}\right)\right\}.$$

The former (latter) equation is decreasing (increasing) with respect to time. Since they satisfy

$$\lim_{t\to 0^-}\frac{\mathrm{d}a^R(0^-,\ t)}{\mathrm{d}t}=n>0,$$
$$\lim_{t\to 0^-}\frac{\mathrm{d}^2a^R(0^-,\ t)}{\mathrm{d}t^2}=-\left\{\frac{(n+\alpha)^2\rho}{\bar{e}}\left[\frac{\alpha\bar{e}}{(n+\alpha)\rho}-\bar{b}\right]+\frac{n\bar{b}}{\alpha}\right\}<0,$$

the signs are determined as follows:

$$\frac{\mathrm{d}a^R(0^-,\ t)}{\mathrm{d}t}\begin{cases}>0 & \text{for }\ \forall t\in[0,\ \hat{t}),\\ =0 & \text{for }\ t=\hat{t},\\ <0 & \text{for }\ \forall t\in(\hat{t},\ \infty),\end{cases}\qquad \frac{\mathrm{d}^2a^R(0^-,\ t)}{\mathrm{d}t^2}\begin{cases}<0 & \text{for }\ \forall t\in[0,\ \acute{t}),\\ =0 & \text{for }\ t=\acute{t},\\ >0 & \text{for }\ \forall t\in(\acute{t},\ \infty),\end{cases}$$

where

$$\hat{t}\equiv\frac{\bar{e}}{(n+\alpha)\rho\bar{b}}\ln\frac{\dfrac{\bar{e}}{\rho}-\bar{b}}{\dfrac{\bar{e}}{\rho}-\left(1+\dfrac{n}{\alpha}\right)\bar{b}}(>0),$$

$$\acute{t}\equiv\frac{\bar{e}}{(n+\alpha)\rho\bar{b}}\ln\frac{\dfrac{\bar{e}}{\rho}-\bar{b}}{\dfrac{\rho}{\bar{e}}\left[\dfrac{\bar{e}}{\rho}-\left(1+\dfrac{n}{\alpha}\right)\bar{b}\right]^2}(>\hat{t}).$$

Therefore, $a^R(0^-,\ t)$ takes a hump-shaped path.

Similarly the dynamics of $a^R(s,\ t)$ for $\forall s\in[0^+,\ \infty)$ is obtained as follows. Substitute (20) into (A. 3) and (A. 5) for $\forall s\in[0^+,\ \infty)$ to have

$$\frac{\mathrm{d}a^R(s,\ t)}{\mathrm{d}t}=\frac{(n+\alpha)^2\rho e^{-\alpha(t-s)}}{n\bar{b}\bar{e}}\left(\frac{\bar{e}}{\rho}-\bar{b}\right)\left\{\frac{\alpha\bar{e}}{(n+\alpha)\rho}-\underbrace{\left[\frac{\alpha\bar{e}}{(n+\alpha)\rho}-\bar{b}\right]}_{(+)}e^{\frac{(n+\alpha)\rho\bar{b}}{\bar{e}}(t-s)}\right\},$$

$$\frac{\mathrm{d}^2a^R(s,\ t)}{\mathrm{d}t^2}=\frac{(n+\alpha)^3\rho^2e^{-\alpha(t-s)}}{n\bar{b}\bar{e}^2}\left(\frac{\bar{e}}{\rho}-\bar{b}\right)\left\{\underbrace{\left[\frac{\alpha\bar{e}}{(n+\alpha)\rho}-\bar{b}\right]^2}_{(+)}e^{\frac{(n+\alpha)\rho\bar{b}}{\bar{e}}(t-s)}-\left[\frac{\alpha\bar{e}}{(n+\alpha)\rho}\right]^2\right\},$$

which gives

$$\lim_{t\to s}\frac{\mathrm{d}a^R(s,\ t)}{\mathrm{d}t}=\frac{(n+\alpha)^2\rho}{n\bar{e}}\left(\frac{\bar{e}}{\rho}-\bar{b}\right)>0,$$
$$\lim_{t\to s}\frac{\mathrm{d}^2a^R(s,\ t)}{\mathrm{d}t^2}=-\frac{(n+\alpha)^3\rho^2}{n\bar{e}^2}\left(\frac{\bar{e}}{\rho}-\bar{b}\right)\left[\frac{2\alpha\bar{e}}{(n+\alpha)\rho}-\bar{b}\right]<0.$$

Therefore, $a^R(s,\ t)$ also follows hump-shaped dynamics:

$$\frac{\mathrm{d}a^R(s,\ t)}{\mathrm{d}t}\begin{cases}>0 & \text{for }\ \forall t\in[s,\ \check{t}),\\ =0 & \text{for }\ t=\check{t},\\ <0 & \text{for }\ \forall t\in(\check{t},\ \infty),\end{cases}\qquad \frac{\mathrm{d}^2a^R(s,\ t)}{\mathrm{d}t^2}\begin{cases}<0 & \text{for }\ \forall t\in[s,\ \grave{t}),\\ =0 & \text{for }\ t=\grave{t},\\ >0 & \text{for }\ \forall t\in(\grave{t},\ \infty),\end{cases}$$

where

$$\check{t}\equiv s+\frac{\bar{e}}{(n+\alpha)\rho\bar{b}}\ln\frac{\dfrac{\alpha\bar{e}}{(n+\alpha)\rho}}{\dfrac{\alpha\bar{e}}{(n+\alpha)\rho}-\bar{b}}(>s),$$

$$\grave{t}\equiv s+\frac{\bar{e}}{(n+\alpha)\rho\bar{b}}\ln\left[\frac{\dfrac{\alpha\bar{e}}{(n+\alpha)\rho}}{\dfrac{\alpha\bar{e}}{(n+\alpha)\rho}-\bar{b}}\right]^2(>\check{t}).$$

Note that $a^R(s, \check{t})$ is more than one. To prove this by contradiction, suppose that $a^R(s, \check{t}) \leq 1$. It is inconsistent with the fact that the social average of financial wealth holdings is $a(t)=b$ because $a^R(0^-, t)$ eventually approaches zero. Consequently, in the case (iii), the relative position of financial wealth holdings once shifts from poor classes to wealthy ones but eventually ends up poor ones, as illustrated in figure 4.

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