

On a Metric Space of Compact Convex Fuzzy Sets with General Set Representation *

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Abstract

In this paper, the author investigates a metric for a class of fuzzy sets, where fuzzy sets are represented by the general set representation approach.

First, based on the general set representation approach, the definition of fuzzy sets is presented from the consistent viewpoint of multivalued logic. Next, a metric for a class of fuzzy sets is introduced. Finally, inspired by the recent excellent researches, especially Krätschmer[1, 2], the properties of the metric space of fuzzy sets are investigated theoretically.

Keywords: fuzzy set, general set representation, metric space of fuzzy sets, multi-valued logic

1 Introduction

The purpose of this paper is to study a metric space for a class of fuzzy sets, which will become a basic tool for defining fuzzy random vectors proposed and intensively investigated by the author[3, 4, 5, 6, 7].

Section 2 is devoted to review the basic properties of the space for non-void compact convex subsets of Euclidean space. The support functions for those subsets are reviewed in Sec. 3, and the metric is introduced by using support functions.

A general set representation approach is adopted in Sec. 4 for defining fuzzy sets, where the viewpoint of the multi-valued logic is persistently maintained. Then, extending the metric defined for the non-void compact convex subsets, that for the space of fuzzy sets is introduced in Sec.5, and its some properties are investigated theoretically.

2 Space of Compact Convex Subsets of $\mathbb{R}^n$

Let $\mathbb{R}^n$ be the n -dimensional Euclidean space. Then, it is well-known that $\mathbb{R}^n$ is a real separable Banach space (complete metric space) with the norm

$$\|x\| = \left[\sum_{i=1}^n x_i^2 \right]^{\frac{1}{2}} \quad (2.1)$$

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for any point $x = (x_1, x_2, \dots, x_n)'$ of $\mathbb{R}^n$. Here, we shall adopt particular notation for certain classes of subsets of $\mathbb{R}^n$ as follows:

$$\mathcal{P}_0(\mathbb{R}^n) = \{ \text{the family of all nonempty subset of } \mathbb{R}^n \}$$

$$\mathcal{K}_{bd}(\mathbb{R}^n) = \{ \text{the family of all non-void bounded subset of } \mathbb{R}^n \}$$

$$\mathcal{K}_{cc}(\mathbb{R}^n) = \{ \text{the family of all non-void compact convex subset of } \mathbb{R}^n \}.$$

Then, Minkowski addition and scalar multiplication between the subsets A and B in $\mathcal{P}_0(\mathbb{R}^n)$ are defined by

$$A + B = \left\{ a + b \mid a \in A, b \in B \right\}$$

$$\lambda \cdot A = \left\{ \lambda \cdot a \mid a \in A \right\},$$

and they are closed in $\mathcal{K}_{cc}(\mathbb{R}^n)$ (see e.g. [8, 9, 10].) The Hausdorff distance between A and B is given by

$$d_H(A, B) = \max \left\{ \sup_{a \in A} \inf_{b \in B} \|a - b\|, \sup_{b \in B} \inf_{a \in A} \|a - b\| \right\}, \quad (2.2)$$

where $\|\cdot\|$ is the Euclidean norm defined by (2.1). Then, it can be shown that the Hausdorff distance has the following properties[9];

$$d_H(A, B) \geq 0 \text{ with } d_H(A, B) = 0 \text{ if and only if } A = B$$

$$d_H(A, B) = d_H(B, A)$$

$$d_H(A, B) \leq d_H(A, C) + d_H(C, B)$$

for A, B and C in $\mathcal{K}_{cc}(\mathbb{R}^n)$. The magnitude of $A \in \mathcal{K}_{cc}(\mathbb{R}^n)$ is defined by

$$\|A\|_H = d_H(A, \{0\}) = \sup_{a \in A} \|a\|. \quad (2.3)$$

Here, $\|A\|_H$ is finite and the supremum in (2.3) is attained when $A \in \mathcal{K}_{cc}(\mathbb{R}^n)$ and it follows that

$$\|\lambda \cdot A\|_H = \lambda \|A\|_H$$

for all $\lambda \geq 0$ and $A \in \mathcal{K}_{cc}(\mathbb{R}^n)$. In addition, we have

$$|\|A\|_H - \|B\|_H| \leq d_H(A, B)$$

for all $A, B \in \mathcal{K}_{cc}(\mathbb{R}^n)$. Thus we can think of the magnitude (2.3) as a Lipschitz function $\|\cdot\|_H : \mathcal{K}_{cc}(\mathbb{R}^n) \rightarrow \mathbb{R}^+$.

Sequences of nested subset in $(\mathcal{K}_{cc}(\mathbb{R}^n), d_H)$ have the following useful intersection and convergence properties[1, 8];

Proposition 2.1. Let $A_i \in \mathcal{K}_{cc}(\mathbb{R}^n) (i = 1, 2, \dots)$ satisfy

$$\dots \subseteq A_i \subseteq \dots \subseteq A_2 \subseteq A_1.$$

Then, we have

$$A = \bigcap_{i=1}^{\infty} A_i \in \mathcal{K}_{cc}(\mathbb{R}^n)$$

and

$$d_H(A_n, A) \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (2.4)$$

3 A Metric Space $K_{cc}(\mathbb{R}^n)$

Let here A be a nonempty subset of $K_{cc}(\mathbb{R}^n)$. Then, the support function of A is defined by

$$\begin{aligned} \text{sp}(x, A) &= \sup \left\{ (x, a) \mid a \in A \right\} \\ &= \sup \left\{ \sum_{i=1}^n x_i \cdot a_i \mid a = (a_1, \dots, a_n)' \in A \right\} \end{aligned}$$

for all $x = (x_1, \dots, x_n)' \in \mathbb{R}^n$, where the supremum is always attained and hence, the support function $\text{sp}(\cdot, A) : \mathbb{R}^n \rightarrow \mathbb{R}$ is well defined. Indeed, it satisfies the bound

$$\begin{aligned} |\text{sp}(x, A)| &= |\sup \{ (x, a) : a \in A \}| \\ &\leq \sup_{a \in A} \|x\| \cdot \|a\| = \|A\|_H \cdot \|x\| \quad \text{for all } x \in \mathbb{R}^n \end{aligned}$$

and is uniformly Lipschitz with

$$|\text{sp}(x, A) - \text{sp}(y, A)| \leq \|A\|_H \cdot \|x - y\| \quad \text{for all } x, y \in \mathbb{R}^n. \quad (3.1)$$

In addition, for all $x \in \mathbb{R}^n, A$ and $B \in K_{cc}(\mathbb{R}^n)$

$$\text{sp}(x, A) \leq \text{sp}(x, B) \quad \text{if } A \subseteq B$$

and

$$\text{sp}(x, A \cup B) \leq \max\{\text{sp}(x, A), \text{sp}(x, B)\}.$$

The support function $\text{sp}(x, A)$ is uniquely paired to the subset in the sense that

$$\text{sp}(x, A) = \text{sp}(x, B) \quad \text{if and only if } A = B$$

for any subsets A and B in $K_{cc}(\mathbb{R}^n)$ (see e.g. [8]). It also preserves set addition and nonnegative scalar multiplication. That is, for all $x \in \mathbb{R}^n$,

$$\text{sp}(x, \lambda \cdot A + \mu \cdot B) = \lambda \cdot \text{sp}(x, A) + \mu \cdot \text{sp}(x, B) \quad (3.2)$$

for $\lambda, \mu \geq 0$ and $A, B \in K_{cc}(\mathbb{R}^n)$, i.e., $\{\text{sp}(\cdot, A) \mid A \in K_{cc}(\mathbb{R}^n)\}$ is a positive convex cone [11]; and, in particular,

$$\text{sp}(x, A + \{y\}) = \text{sp}(x, A) + (x, y) \quad (3.3)$$

for any $y \in \mathbb{R}^n$;

$$\text{sp}(x, \lambda \cdot A) = \lambda \cdot \text{sp}(x, A) \quad (3.4)$$

$$\text{sp}(\lambda \cdot x, A) = \lambda \cdot \text{sp}(x, A) \quad (3.5)$$

for any $\lambda \geq 0$; and subadditive, i.e.,

$$\text{sp}(x + y, A) \leq \text{sp}(x, A) + \text{sp}(y, A) \quad (3.6)$$

for all $x, y \in \mathbb{R}^n$. Moreover, combining (3.5) and (3.6) we see that $\text{sp}(\cdot, A)$ is a convex function, that is, it satisfies

$$\text{sp}(\lambda x + (1 - \lambda)y, A) \leq \lambda \cdot \text{sp}(x, A) + (1 - \lambda)\text{sp}(y, A)$$

for all $x, y \in \mathbb{R}^n$ and $\lambda \in [0, 1]$.

It is also known that the Hausdorff metric given in (2.2) is related to the support function for subsets $A, B \in K_{cc}(\mathbb{R}^n)$ by

$$d_H(A, B) = \sup \left\{ |\text{sp}(x, A) - \text{sp}(x, B)| \mid x \in S^{n-1} \right\} \quad (3.7)$$

where $S^{n-1} = \{x \in \mathbb{R}^n : \|x\| = 1\}$ is the unit sphere in $\mathbb{R}^n$ (see e.g. [12]).

Let $(S^{n-1}, \mathcal{B}(S^{n-1}), \mu_{S^{n-1}})$ be the Lebesgue measure space, where $\mathcal{B}(S^{n-1})$ is the σ -algebra on S^{n-1} and $\mu_{S^{n-1}}$ be the Lebesgue measure on S^{n-1} ($\mu_{S^{n-1}}(S^{n-1}) = 1$). Since the support function $\text{sp}(x, A)$ for any subset $A \in K_{cc}(\mathbb{R}^n)$ satisfies the Lipschitz condition (3.1), it is continuous with respect to $x \in S^{n-1}$ and hence $\text{sp}(x, \cdot)$ is $\mathcal{B}(S^{n-1})$ -measurable. Then, the Steiner centroid $\sigma(A)$ of an A is defined by

$$\sigma(A) = n \cdot \int_{S^{n-1}} x \cdot \text{sp}(x, A) d\mu_{S^{n-1}}(x), \quad (3.8)$$

where the existence of (3.8) is guaranteed by the boundedness of $|x \cdot \text{sp}(x, A)| \leq \|A\|_H < +\infty$ (see e.g. [8] and [13]). For any $A \in K_{cc}(\mathbb{R}^n)$ the Steiner centroid $\sigma(A)$ satisfies [14]

$$\sigma(A) \in A \quad \text{and} \quad \sigma(-A) = -\sigma(A),$$

and since $\text{sp}(-x, -A) = \text{sp}(x, A)$, it follows that $\sigma(-A) = -\sigma(A)$. Hence, if A is symmetric with respect to the origin, that is if $-A = A$, we have $\sigma(A) = 0$. Moreover, when it is considered as a mapping $\sigma : K_{cc}(\mathbb{R}^n) \rightarrow \mathbb{R}^n$, the Steiner centroid is linear (see e.g. [14])

$$\sigma(\alpha A + \beta B) = \alpha \sigma(A) + \beta \sigma(B)$$

for all $A, B \in K_{cc}(\mathbb{R}^n)$ and $\alpha, \beta \in \mathbb{R}$, and Lipschitz continuous with Lipschitz constant n ;

$$\|\sigma(A) - \sigma(B)\| \leq n \cdot d_H(A, B).$$

And also, for each convex set, the set $A^0 = A - \sigma(A)$ has its Steiner centroid at the origin, i.e.

$$\sigma(A^0) = \sigma(A - \sigma(A)) = 0.$$

The Steiner centroid is a very useful tool in the theory of convex sets.

Every support function $\text{sp}(\cdot, \cdot)$ of a non-void convex compact subset of $\mathbb{R}^n$ is continuous and bounded, in particular $\mu_{S^{n-1}}$ -integrable of order $p \in [1, +\infty)$ and bounded with respect to $\mu_{S^{n-1}}$, which enables us to define the metric as follows;

$$\delta_p(A, B) = \left(\int_{S^{n-1}} |\text{sp}(x, A) - \text{sp}(x, B)|^p d\mu_{S^{n-1}}(x) \right)^{\frac{1}{p}} \quad (3.9)$$

for any $A, B \in K_{cc}(\mathbb{R}^n)$ and the some fixed $p \in [1, +\infty)$, where the existence of (3.9) is guaranteed by the boundedness of $\text{sp}(x, A)$ and $\text{sp}(x, B)$. From (3.7) we immediately have

$$\delta_p(A, B) \leq d_H(A, B)$$

for all $A, B \in K_{cc}(\mathbb{R}^n)$. Furthermore, it is known that

$$d_H(A, B) \leq 2\delta_p(A, B)$$

if $n = 1$, and there exists some positive constant C_{np} such that

$$d_H(A, \{0\}) \leq C_{np}\delta_p(A, \{0\}) \quad (3.10)$$

for arbitrary $A \in K_{cc}(\mathbb{R}^n)$ (see [15]).

4 Fuzzy Sets with General Set Representation Approach

In this paper, fuzzy sets in $\mathbb{R}^n$ are defined by using the general set representation approach. A fuzzy set $\tilde{U}$ as the vague perception of $u_o \in \mathbb{R}^n$ is defined by the triple

$$\tilde{U} = (\mathbb{R}^n, [\tilde{U}], s_{\tilde{U}}) \quad (4.1)$$

with

$$[\tilde{U}] = \left\{ [\tilde{U}]_\alpha \mid \alpha \in I \right\}, \quad (4.2)$$

where $\mathbb{R}^n$ is the n -dimensional Euclidean space called the basic space; $s_{\tilde{U}}$ is the predicate, i.e., $s_{\tilde{U}} : \mathbb{R}^n \rightarrow \mathcal{P}$ with $\mathcal{P}$ the “universe of discourse” defined by a set of statements, which assigns a proposition

$$s_{\tilde{U}}(x) = \left\{ x \text{ coincides with } u_o \right\} \quad (4.3)$$

to each element $x \in \mathbb{R}^n$; and $[\tilde{U}] = \{[\tilde{U}]_\alpha \mid \alpha \in I\}$ with $I = (0, 1]$ is the family of subsets of $\mathbb{R}^n$ satisfying

$$L_\alpha \tilde{U} \subseteq [\tilde{U}]_\alpha \subseteq L_{\bar{\alpha}} \tilde{U} \quad \text{for any } \alpha \in I,$$

where $L_\alpha \tilde{U}$ and $L_{\bar{\alpha}} \tilde{U}$ are the strong cut(strong level set) and the level set of $\tilde{U}$ at the level α defined respectively by

$$L_\alpha \tilde{U} = \left\{ x \mid (x \in \mathbb{R}^n) \wedge ((\tilde{U})(x) > \alpha) \right\}$$

and

$$L_{\bar{\alpha}} \tilde{U} = \left\{ x \mid (x \in \mathbb{R}^n) \wedge ((\tilde{U})(x) \geq \alpha) \right\},$$

where $(\tilde{U})(x)$ is the membership function of $\tilde{U}$ given by

$$(\tilde{U})(x) = t(s_{\tilde{U}}(x)) \quad (4.4)$$

and $t(*)$ in (4.4) is the truth function of $*$ in the sense of multivalued logic[16]. The crisp point u_o in (4.3), the vague perception of which gives the fuzzy set $\tilde{U}$, is called the original point of $\tilde{U}$.

The set representation of a fuzzy set (4.2) satisfies

$$0 < \alpha < \beta < 1 \Rightarrow [\tilde{U}]_\beta \subseteq [\tilde{U}]_\alpha \subseteq \mathbb{R}^n,$$

and

$$(\tilde{U})(x) = \sup \left\{ \alpha \mid (\alpha \in I) \wedge (x \in [\tilde{U}]_\alpha) \right\}.$$

The families of the level sets and the strong cuts, i.e., $\{L_{\tilde{\alpha}}\tilde{U} \mid \alpha \in I\}$ and $\{L_{\alpha}\tilde{U} \mid \alpha \in I\}$ are themselves the set representations of $\tilde{U}$, and they are the lower and the upper bound of set representations of $\tilde{U}$. It can be also shown that [17, 18]

$$L_{\alpha}\tilde{U} = \bigcup_{\beta \in (\alpha, 1)} [\tilde{U}]_{\beta} \quad \text{for } \alpha \in [0, 1)$$

and

$$L_{\tilde{\alpha}}\tilde{U} = \bigcap_{\beta \in (0, \alpha)} [\tilde{U}]_{\beta} \quad \text{for } \alpha \in (0, 1].$$

In this paper, we restrict our attention to the following family of fuzzy sets.

Definition 4.1. The family of fuzzy sets is denoted by $F(\mathbb{R}^n)$, whose element $\tilde{U}$ satisfies the following conditions:

- (i) The closed convex hull $\overline{\text{co}}[\tilde{U}]_{\alpha}$, where $[\tilde{U}]_{\alpha}$ is any element of the set representation $[\tilde{U}]$ of a fuzzy set $\tilde{U}$, is compact subset of $\mathbb{R}^n$;
- (ii) $\overline{\text{co}}[\tilde{U}]_{\alpha}$ ($\alpha \in I$) is piecewise left-continuous with respect to the Hausdorff metric d_H , i.e.,

$$\lim_{\beta \uparrow \alpha} d_H(\overline{\text{co}}[\tilde{U}]_{\beta}, \overline{\text{co}}[\tilde{U}]_{\alpha}) = 0 \quad (4.5)$$

except for some finite points $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_m < 1$ of I .

- (iii) The original vector u_o of $\tilde{U}$ satisfies

$$u_o \in \text{supp.}\tilde{U}, \quad (4.6)$$

where $\text{supp.}\tilde{U}$ is the support of $\tilde{U}$ defined by

$$\text{supp.}\tilde{U} = \text{cl.}\left(\bigcup_{\alpha \in I} \overline{\text{co}}[\tilde{U}]_{\alpha}\right). \quad (4.7)$$

Furthermore, $F_{\text{bd}}(\mathbb{R}^n)$ is the family of all fuzzy sets in $F(\mathbb{R}^n)$ such that $\text{supp.}\tilde{U}$ is compact.

5 A Metric Space for a Class of Fuzzy Sets

The concept of support function for a nonempty compact convex subset of $\mathbb{R}^n$, introduced in Sec.2, can be usefully generalized to the fuzzy sets in $F(\mathbb{R}^n)$. The support function of the fuzzy set $\tilde{U} \in F(\mathbb{R}^n)$ is given by

$$\varphi_{\tilde{U}}(\alpha, x) = \begin{cases} 0 & \text{for } \alpha = 0 \\ \text{sp}(x, \overline{\text{co}}[\tilde{U}]_{\alpha}) = \sup \left\{ (x, u) \mid u \in \overline{\text{co}}[\tilde{U}]_{\alpha} \right\} & \text{for } \alpha \in (0, 1]. \end{cases} \quad (5.1)$$

It can be easily shown from (3.5) and (3.6) that the support function $\varphi_{\tilde{U}}$ is continuous with respect to $x \in S^{n-1}$, positive homogeneous, i.e.,

$$\varphi_{\tilde{U}}(\alpha, \lambda \cdot x) = \lambda \cdot \varphi_{\tilde{U}}(\alpha, x) \quad \text{for } \lambda \geq 0, x \in S^{n-1} \text{ at each } \alpha \in [0, 1],$$

subadditive, i.e.,

$$\varphi_{\tilde{U}}(\alpha, x_1 + x_2) \leq \varphi_{\tilde{U}}(\alpha, x_1) + \varphi_{\tilde{U}}(\alpha, x_2) \quad \text{for } x_1, x_2 \in S^{n-1}$$

at each $\alpha \in [0, 1]$; and nonincreasing, i.e.,

$$\varphi_{\tilde{U}}(\alpha, x) \geq \varphi_{\tilde{U}}(\beta, x)$$

at each $x \in S^{n-1}$, where $0 \leq \alpha \leq \beta \leq 1$.

Let $(\mathbb{R}, \mathcal{B}, \mu)$ and $([0, 1], \mathcal{B}_{[0,1]}, \mu_{[0,1]})$ be the measure spaces, where $\mathcal{B}$ and $\mathcal{B}_{[0,1]}$ are the σ -algebras on $\mathbb{R}$ and $[0, 1]$, respectively, and μ and $\mu_{[0,1]}$ are their Lebesgue measures. Then, the following property is confirmed:

Proposition 5.1. Let $\tilde{U} \in F(\mathbb{R}^n)$. Then, $\varphi_{\tilde{U}}(\cdot, \cdot)$ is $\mathcal{B}_{[0,1]} \otimes \mathcal{B}(S^{n-1})$ -measurable, and if $\tilde{U} \in F_{\text{bd}}(\mathbb{R}^n)$, then $\varphi_{\tilde{U}}$ is integrable of order p for every $p \in [1, +\infty)$.

Proof. It can be shown from (5.1) and (3.7) that

$$\begin{aligned} |\varphi_{\tilde{U}}(\alpha, x) - \varphi_{\tilde{U}}(\beta, x)| &= |\text{sp}(x, \overline{\text{co}}[\tilde{U}]_{\alpha}) - \text{sp}(x, \overline{\text{co}}[\tilde{U}]_{\beta})| \\ &\leq d_H(\overline{\text{co}}[\tilde{U}]_{\alpha}, \overline{\text{co}}[\tilde{U}]_{\beta}) \end{aligned}$$

hold for $\alpha, \beta \in I$ and $x \in S^{n-1}$. Since $\overline{\text{co}}[\tilde{U}]_{\alpha}$ ($\alpha \in I$) is piecewise left-continuous with respect to the Hausdorff metric d_H , it follows

$$\lim_{\beta \uparrow \alpha} |\varphi_{\tilde{U}}(\alpha, x) - \varphi_{\tilde{U}}(\beta, x)| \leq \lim_{\beta \uparrow \alpha} d_H(\overline{\text{co}}[\tilde{U}]_{\alpha}, \overline{\text{co}}[\tilde{U}]_{\beta}) = 0 \quad (5.2)$$

except for $\alpha = \alpha_1, \alpha_2, \dots, \alpha_m$. This means that $\varphi_{\tilde{U}}(\alpha, x)$ is also piecewise left-continuous on $\alpha \in I$ for every $x \in S^{n-1}$, and with its non-increasing property we can conclude that $\varphi(\alpha, x)$ is piecewise upper semi-continuous on α for every $x \in S^{n-1}$.

With the piecewise upper semi-continuity of $\varphi_{\tilde{U}}$ shown by (5.2), we know that $\varphi_{\tilde{U}}(\cdot, x)$ is $\mathcal{B}_{[0,1]}$ -measurable for any fixed $x \in S^{n-1}$. The support function $\varphi_{\tilde{U}}(\alpha, x)$ is obviously continuous with respect to x for every α . Then, we can conclude that $\varphi_{\tilde{U}}(\cdot, \cdot)$ is $\mathcal{B}_{[0,1]} \otimes \mathcal{B}(S^{n-1})$ -measurable[19]. Furthermore, from the condition of $F_{\text{bd}}(\mathbb{R}^n)$, if $\tilde{U} \in F_{\text{bd}}(\mathbb{R}^n)$, it follows

$$|\varphi_{\tilde{U}}(\alpha, x)|^p \leq d_H(\overline{\text{co}}[\tilde{U}]_{\alpha}, \{0\})^p \leq d_H(\text{supp. } \tilde{U}, \{0\})^p < +\infty. \quad (5.3)$$

Hence,

$$\int |\varphi_{\tilde{U}}(\alpha, x)|^p d\mu_{[0,1]}(\alpha) \otimes \mu_{S^{n-1}}(x) < +\infty, \quad (5.4)$$

which means that $\varphi_{\tilde{U}}$ is integrable of order p . $\square$

We denote the family of fuzzy sets by $F^p(\mathbb{R}^n)$, each element of which is a member of $F(\mathbb{R}^n)$ and its support function $\varphi_{\tilde{U}}$ is integrable of the order p with respect to $\mu_{[0,1]} \otimes \mu_{S^{n-1}}$. From (5.3), it is clear that

$$F_{\text{bd}}(\mathbb{R}) \subset F^p(\mathbb{R}^n)$$

for any $p \in [1, +\infty)$. For $\lambda, \mu \geq 0$ and $\tilde{U}, \tilde{V} \in F(\mathbb{R}^n)$, we have

$$\varphi_{\lambda \cdot \tilde{U} + \mu \cdot \tilde{V}}(\alpha, x) = \begin{cases} 0 & \text{for } \alpha = 0 \\ \text{sp}(x, \overline{\text{co}}[\lambda \cdot \tilde{U} + \mu \cdot \tilde{V}]_{\alpha}) & \text{for } \alpha \in I. \end{cases}$$

Since it can be shown that (see e.g. [17, 18])

$$[\lambda \cdot \tilde{U} + \mu \cdot \tilde{V}]_\alpha = \lambda \cdot [\tilde{U}]_\alpha + \mu \cdot [\tilde{V}]_\alpha,$$

and (see e.g. [9])

$$\overline{\text{co}}(\lambda \cdot [\tilde{U}]_\alpha + \mu \cdot [\tilde{V}]_\alpha) = \lambda \cdot \overline{\text{co}}[\tilde{U}]_\alpha + \mu \cdot \overline{\text{co}}[\tilde{V}]_\alpha$$

for any $\alpha \in I$ and $\lambda, \mu \in \mathbb{R}$. Then, using (3.2), it follows

$$\Phi_{\lambda \cdot \tilde{U} + \mu \cdot \tilde{V}}(\alpha, x) = \lambda \cdot \Phi_{\tilde{U}} + \mu \cdot \Phi_{\tilde{V}} \quad \text{for any } \alpha \in I, x \in S^{n-1} \text{ and } \lambda, \mu \geq 0. \quad (5.5)$$

Then, we know that the mapping $\tilde{U} \mapsto \Phi_{\tilde{U}}$ is an isomorphism of $F(\mathbb{R}^n)$ onto a positive convex cone of $\mathcal{B}_{[0,1]} \otimes \mathcal{B}(S^{n-1})$ -measurable functions, preserving the semi-linear structure (5.5)[11].

Applying Fubini's theorem (see e.g. [20]), we can write

$$\int (\Phi_{\tilde{U}}(\alpha, x))^p d\mu_{[0,1]}(\alpha) \otimes \mu_{S^{n-1}}(x) = \int_0^1 \int_{S^{n-1}} (\Phi_{\tilde{U}}(\alpha, x))^p d\mu_{[0,1]}(\alpha) d\mu_{S^{n-1}}(x)$$

for any $\tilde{U} \in F^p(\mathbb{R}^n)$ and $p \in [1, +\infty)$. Then, the quantity $\rho_p(\tilde{U}, \tilde{V})$ for any $\tilde{U}, \tilde{V} \in F^p(\mathbb{R}^n)$ is defined by

$$\begin{aligned} \rho_p(\tilde{U}, \tilde{V}) &= \left(\int |\Phi_{\tilde{U}}(\alpha, x) - \Phi_{\tilde{V}}(\alpha, x)|^p d\mu_{[0,1]}(\alpha) \otimes \mu_{S^{n-1}}(x) \right)^{\frac{1}{p}} \\ &= \left(\int_0^1 \int_{S^{n-1}} |\Phi_{\tilde{U}}(\alpha, x) - \Phi_{\tilde{V}}(\alpha, x)|^p d\mu_{[0,1]}(\alpha) d\mu_{S^{n-1}}(x) \right)^{\frac{1}{p}} \\ &= \left(\int_0^1 \delta_p(\overline{\text{co}}[\tilde{U}]_\alpha, \overline{\text{co}}[\tilde{V}]_\alpha)^p d\mu_{[0,1]}(\alpha) \right)^{\frac{1}{p}} \end{aligned}$$

where (3.9) has been used. Then, the symmetry of ρ_p is obvious, $\rho_p(\tilde{U}, \tilde{U}) = 0$ for every $\tilde{U} \in F^p(\mathbb{R}^n)$, and the triangle inequality is proved by Minkowski inequality, i.e.,

$$\begin{aligned} \rho_p(\tilde{U}, \tilde{V}) &= \left(\int_0^1 \int_{S^{n-1}} |\Phi_{\tilde{U}}(\alpha, x) - \Phi_{\tilde{V}}(\alpha, x)|^p d\mu_{[0,1]}(\alpha) d\mu_{S^{n-1}}(x) \right)^{\frac{1}{p}} \\ &\leq \left(\int_0^1 \int_{S^{n-1}} (|\Phi_{\tilde{U}}(\alpha, x) - \Phi_{\tilde{W}}(\alpha, x)| + |\Phi_{\tilde{W}}(\alpha, x) - \Phi_{\tilde{V}}(\alpha, x)|)^p d\mu_{[0,1]}(\alpha) d\mu_{S^{n-1}}(x) \right)^{\frac{1}{p}} \\ &\leq \left(\int_0^1 \int_{S^{n-1}} |\Phi_{\tilde{U}}(\alpha, x) - \Phi_{\tilde{W}}(\alpha, x)|^p d\mu_{[0,1]}(\alpha) d\mu_{S^{n-1}}(x) \right)^{\frac{1}{p}} \\ &\quad + \left(\int_0^1 \int_{S^{n-1}} |\Phi_{\tilde{W}}(\alpha, x) - \Phi_{\tilde{V}}(\alpha, x)|^p d\mu_{[0,1]}(\alpha) d\mu_{S^{n-1}}(x) \right)^{\frac{1}{p}} \\ &= \rho_p(\tilde{U}, \tilde{W}) + \rho_p(\tilde{W}, \tilde{V}) \end{aligned}$$

for any $\tilde{U}, \tilde{V}, \tilde{W} \in F^p(\mathbb{R}^n)$. Furthermore, let $\tilde{U}, \tilde{V} \in F^p(\mathbb{R}^n)$ with $\rho_p(\tilde{U}, \tilde{V}) = 0$. Then, we have

$$\begin{aligned} \Phi_{\tilde{U}}(\alpha, x) &= \Phi_{\text{cl.}\tilde{U}}(\alpha, x) = \Phi_{\text{co}\tilde{U}}(\alpha, x) = \Phi_{\overline{\text{co}}\tilde{U}}(\alpha, x) \\ &= \Phi_{\tilde{V}}(\alpha, x) = \Phi_{\text{cl.}\tilde{V}}(\alpha, x) = \Phi_{\text{co}\tilde{V}}(\alpha, x) = \Phi_{\overline{\text{co}}\tilde{V}}(\alpha, x) \quad \text{a.s. w.r.t. } \mu_{[0,1]} \otimes \mu_{S^{n-1}}, \end{aligned}$$

and this is a equivalence relation $\sim$. Hence, we can define the quotient set as follows:

$$\mathbb{F}^p(\mathbb{R}^n) = \mathbb{F}^p(\mathbb{R}^n) / \sim.$$

Then, $(\mathbb{F}^p(\mathbb{R}^n), \rho_p)$ is a metric space. The quotient metric space $\mathbb{F}_{\text{bd}}(\mathbb{R}^n) / \sim$ is also defined by

$$\mathbb{F}_{\text{bd}}(\mathbb{R}^n) = \mathbb{F}_{\text{bd}}(\mathbb{R}^n) / \sim.$$

Proposition 5.2. $\mathbb{F}_{\text{bd}}(\mathbb{R}^n)$ is a dense subset of $\mathbb{F}^p(\mathbb{R}^n)$ for any $p \in [1, +\infty)$.

Proof. Let $\tilde{U} \in \mathbb{F}^p(\mathbb{R}^n)$ and $\{\tilde{U}_m \in \mathbb{F}_{\text{bd}}(\mathbb{R}^n) | m = 1, 2, \dots\}$ be a sequence of fuzzy sets, where the set representation of $\tilde{U}_m$ consists of

$$[\tilde{U}_m]_\alpha = [\tilde{U}]_{\frac{1}{m}} \quad \text{for } 0 < \alpha \leq \frac{1}{m}$$

and

$$[\tilde{U}_m]_\alpha = [\tilde{U}]_\alpha \quad \text{for } \frac{1}{m} < \alpha \leq 1.$$

Then, using (3.7) we can evaluate $\rho_p(\tilde{U}, \tilde{U}_m)$ as follows:

$$\begin{aligned} \rho_p(\tilde{U}, \tilde{U}_m)^p &= \int_0^1 \int_{S^{n-1}} |\varphi_{\tilde{U}}(\alpha, x) - \varphi_{\tilde{U}_m}(\alpha, x)|^p d\mu_{[0,1]}(\alpha) d\mu_{S^{n-1}}(x) \\ &< \int_0^1 h_{\tilde{U}, \tilde{U}_m}(\alpha)^p d\mu_{[0,1]}(\alpha), \end{aligned} \quad (5.6)$$

where

$$h_{\tilde{U}, \tilde{U}_m}(\alpha) = \begin{cases} d_H(\overline{\text{co}}[\tilde{U}]_\alpha, \overline{\text{co}}[\tilde{U}_m]_\alpha) & \text{for } 0 < \alpha \leq 1, \\ 0 & \text{for } \alpha = 0. \end{cases}$$

Let $\{\tilde{V}_m | m = 1, 2, \dots\}$ be a sequence of fuzzy sets in $\mathbb{F}^p(\mathbb{R}^n)$ with the set representation such as

$$[\tilde{V}_m]_\alpha = [\tilde{U}]_{\frac{1}{m}} \quad \text{for } 0 < \alpha \leq 1.$$

Then, from (5.6), it follows that

$$\rho_p(\tilde{U}, \tilde{U}_m)^p < \int_0^1 1_{[0, \frac{1}{m}]} h_{\tilde{U}, \tilde{V}_m}(\alpha)^p d\mu_{[0,1]}(\alpha).$$

Since $\overline{\text{co}}[\tilde{V}]_{\frac{1}{m}}$ is a subset of $\overline{\text{co}}[\tilde{U}]_\alpha$ for every $\alpha \in (0, \frac{1}{m}]$, we obtain

$$|1_{[0, \frac{1}{m}]} h_{\tilde{U}, \tilde{V}_m}(\alpha)|^p \leq |2h_{\tilde{U}, \{0\}}(\alpha)|^p \leq C_p \delta_p(\tilde{U}, \{0\})^p,$$

where C_p is a some positive constant determined by (3.10) and $\delta_p(\tilde{U}, \{0\})^p$ is integrable with respect to $\mu_{[0,1]}$. Furthermore,

$$\lim_{m \rightarrow \infty} 1_{[0, \frac{1}{m}]}(\alpha) h_{\tilde{U}, \tilde{V}_m}(\alpha) = 0 \quad \text{for } \alpha \in I.$$

Hence, applying Lebesgue dominated convergence theorem(see e.g. [20]), we can conclude

$$\lim_{m \rightarrow \infty} \rho_p(\tilde{U}, \tilde{U}_m) = 0.$$

□

Proposition 5.3. $\mathbb{F}_{\text{bd}}(\mathbb{R}^n)$ is a separable metric space for any $p \in [1, +\infty)$.

Proof. Taking any $\tilde{U} \in \mathbb{F}_{\text{bd}}(\mathbb{R}^n)$ and $\varepsilon > 0$. Since $\text{supp.}\tilde{U}$ is compact, there exists a minimal cover $\{E_i\}$ of cubes

$$E_i = \prod_{j=1}^n [a_{ij}, b_{ij}), \quad i = 1, 2, \dots, r$$

with a_{ij}, b_{ij} rational,

$$0 < b_{ij} - a_{ij} < \frac{\varepsilon}{4\sqrt{n}} \quad \text{and} \quad \text{supp.}\tilde{U} \subseteq \bigcup_{i=1}^n E_i.$$

From the compact property of $\overline{\text{co}}[\tilde{U}]_\alpha$ for each $\alpha \in I$, $\overline{\text{co}}[\tilde{U}]_\alpha$ has a minimal subcover $\{E_{i(\alpha)}\}$ of $\{E_i\}$ such that

$$\overline{\text{co}}[\tilde{U}]_\alpha \subseteq \bigcup_{i(\alpha)} E_i(\alpha).$$

Write

$$V_k = (\text{supp.}\tilde{U}) \cap \left(\bigcup_{i=k}^r E_i \right).$$

Note that

$$d_H(\text{supp.}\tilde{U}, \bigcup_{i=1}^r E_i) \leq \frac{\varepsilon}{4}, \quad d_H(\overline{\text{co}}[\tilde{U}]_\alpha, \bigcup_{i(\alpha)} E_{i(\alpha)}) \leq \frac{\varepsilon}{4}$$

and also that

$$d_H(V_k, \bigcup_{i=k}^r E_i) \leq \frac{\varepsilon}{4}.$$

Set here

$$\alpha_i = \sup \left\{ \alpha \mid (\overline{\text{co}}[\tilde{U}]_\alpha) \cap (\text{cl.} E_i) \neq \emptyset, \alpha \in I \right\}$$

and relabel $E_1, E_2, \dots, E_r$ so that $0 = \alpha_0 \leq \alpha_1 \leq \dots \leq \alpha_r = 1$. Define the fuzzy set $\tilde{\Phi}$ whose set representation is given through

$$[\tilde{\Phi}]_\alpha = \bigcup_{i=k}^r E_i \quad \text{for} \quad \alpha_{k-1} < \alpha \leq \alpha_k, \quad k = 1, 2, \dots, r.$$

For any $\alpha \in I$, there exists $1 \leq k \leq r$ so that $\alpha_{k-1} \leq \alpha \leq \alpha_k$. If k is the largest integer such that

$$(\text{supp.}\tilde{U})^c \cap E_i \neq \emptyset$$

for $i = 1, 2, \dots, k$, then

$$d_H(\overline{\text{co}}[\tilde{U}]_\alpha, [\tilde{\Phi}]_\alpha) \leq \frac{\varepsilon}{4}.$$

If $\alpha_{k-1} = \alpha < \alpha_k$,

$$d_H(\overline{\text{co}}[\tilde{U}]_\alpha, [\tilde{\Phi}]_\alpha) = d_H([\tilde{U}]_{\alpha_{k-1}}, \bigcup_{i=k}^r E_i) \leq \frac{\varepsilon}{4},$$

and similarly if $\alpha_{k-1} < \alpha = \alpha_k$,

$$d_H(\overline{\text{co}}[\tilde{U}]_\alpha, [\tilde{\Phi}]_\alpha) = d_H(\overline{\text{co}}[\tilde{U}]_{\alpha_k}, \bigcup_{i=k}^r E_i) \leq \frac{\varepsilon}{4}.$$

For $\alpha_{k-1} < \alpha < \alpha_k$,

$$d_H(\overline{\text{co}}[\tilde{U}]_\alpha, [\tilde{\Phi}]_\alpha) = d_H(\overline{\text{co}}[\tilde{U}]_{\alpha_k}, \bigcup_{i=k}^r E_i) \leq 2 \max_{1 \leq i \leq r} \text{diam}(E_i) \leq \frac{\varepsilon}{2}.$$

Furthermore, we have

$$d_H([\tilde{\Phi}]_\alpha, \overline{\text{co}}[\tilde{\Phi}]_\alpha) \leq \max_{1 \leq k \leq r} d_H(\bigcup_{i=k}^r E_i, \overline{\text{co}}(\bigcup_{i=k}^r E_i)) \leq \frac{\varepsilon}{4}.$$

Thus

$$\begin{aligned} \rho_p(\tilde{U}, \tilde{\Phi}) &= \left(\int_0^1 \int_{S^{n-1}} |\varphi_{\tilde{U}}(\alpha, x) - \varphi_{\tilde{\Phi}}(\alpha, x)|^p d\mu_{[0,1]}(\alpha) d\mu_{S^{n-1}}(x) \right)^{\frac{1}{p}} \\ &\leq \left(\int_0^1 d_H(\overline{\text{co}}[\tilde{U}]_\alpha, \overline{\text{co}}[\tilde{\Phi}]_\alpha)^p d\mu_{[0,1]}(\alpha) \right)^{\frac{1}{p}} \\ &\leq \left(\int_0^1 \left(d_H(\overline{\text{co}}[\tilde{U}]_\alpha, [\tilde{\Phi}]_\alpha) + d_H([\tilde{\Phi}]_\alpha, \overline{\text{co}}[\tilde{\Phi}]_\alpha) \right)^p d\mu_{[0,1]}(\alpha) \right)^{\frac{1}{p}} \\ &\leq \left(\int_0^1 d_H(\overline{\text{co}}[\tilde{U}]_\alpha, [\tilde{\Phi}]_\alpha)^p d\mu_{[0,1]}(\alpha) \right)^{\frac{1}{p}} \\ &\quad + \left(\int_0^1 d_H([\tilde{\Phi}]_\alpha, \overline{\text{co}}[\tilde{\Phi}]_\alpha)^p d\mu_{[0,1]}(\alpha) \right)^{\frac{1}{p}} \\ &\leq \frac{3\varepsilon}{4}. \end{aligned}$$

Now let

$$M > 4(r-1) \text{diam}(\bigcup_{i=1}^s E_i).$$

For $i = 1, 2, \dots, r$, relabel $\alpha_1, \alpha_2, \dots, \alpha_r$ and exclude duplicates if necessary, so that

$$0 \leq \alpha_0 < \alpha_1 < \alpha_2 < \dots < \alpha_s = 1$$

with $s \leq r$. If α_i is irrational, choose β_i rational such that

$$\max \left\{ \alpha_{i-1}, \alpha_i - \frac{\varepsilon^p}{M^p} \right\} < \beta_i < \alpha_i$$

and if α_i is rational, set $\beta_i = \alpha_i$. Define $\tilde{\Psi} \in \mathbb{F}_{\text{bd}}(\mathbb{R}^n)$ by its set representation $[\tilde{\Psi}]$ with

$$[\tilde{\Psi}]_\beta = \bigcup_{i=k}^s E_i \quad \text{for } \beta_{k-1} < \beta \leq \beta_k, \quad k = 1, 2, \dots, s.$$

Then, it follows

$$\begin{aligned}
 \rho_p(\tilde{\Phi}, \tilde{\Psi}) &= \left(\int_0^1 \int_{S^{n-1}} |\varphi_{\tilde{\Phi}}(\alpha, x) - \varphi_{\tilde{\Psi}}(\alpha, x)|^p d\mu_{[0,1]}(\alpha) d\mu_{S^{n-1}}(x) \right)^{\frac{1}{p}} \\
 &\leq \left(\int_0^1 d_H(\overline{\text{co}}[\tilde{\Phi}]_\alpha, \overline{\text{co}}[\tilde{\Psi}]_\alpha)^p d\mu_{[0,1]}(\alpha) \right)^{\frac{1}{p}} \\
 &\leq \left(\sum_{i=1}^{s-1} \int_{\beta_i}^{\alpha_i} \left(d_H(\overline{\text{co}}[\tilde{\Phi}]_\alpha, \overline{\text{co}}[\tilde{\Psi}]_\alpha) \right)^p d\mu_{[0,1]}(\alpha) \right)^{\frac{1}{p}} \\
 &\leq \text{diam}\left(\bigcup_{i=1}^s E_i\right) \left(\sum_{i=1}^{s-1} (\alpha_i - \beta_i) \right)^{\frac{1}{p}} \\
 &\leq \frac{\varepsilon}{4}.
 \end{aligned}$$

Finally, we have

$$\rho_p(\tilde{U}, \tilde{\Psi}) \leq \rho_p(\tilde{U}, \tilde{\Phi}) + \rho_p(\tilde{\Phi}, \tilde{\Psi}) \leq \frac{3\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon$$

and the result follows. $\square$

For every $p \in [1, +\infty)$, we can embed $\mathbb{F}^p(\mathbb{R}^n)$ isomorphically into the $\mathbb{L}^p$ -space

$$j_{\mathbb{F}^p(\mathbb{R}^n)} : \mathbb{F}^p(\mathbb{R}^n) \rightarrow \mathbb{L}^p([0, 1] \times S^{n-1}), \quad j_{\mathbb{F}^p(\mathbb{R}^n)}(\tilde{U}) = \varphi_{\tilde{U}}, \quad (5.7)$$

satisfying

1. $j_{\mathbb{F}^p(\mathbb{R}^n)}$ is injective,

2.

$$j_{\mathbb{F}^p(\mathbb{R}^n)}(\tilde{U} + \tilde{V}) = j_{\mathbb{F}^p(\mathbb{R}^n)}(\tilde{U}) + j_{\mathbb{F}^p(\mathbb{R}^n)}(\tilde{V})$$

for $\tilde{U}, \tilde{V} \in \mathbb{F}^p(\mathbb{R}^n)$, and

3.

$$j_{\mathbb{F}^p(\mathbb{R}^n)}(\lambda \cdot \tilde{U}) = \lambda \cdot j_{\mathbb{F}^p(\mathbb{R}^n)}(\tilde{U})$$

for $\tilde{U} \in \mathbb{F}^p(\mathbb{R}^n)$ and $\lambda \geq 0$.

Proposition 5.4. For every $p \in [1, +\infty)$, $(\mathbb{F}^p(\mathbb{R}^n), \rho_p)$ is a complete separable metric space.

Proof. Propositions 5.2 and 5.3 show that $\mathbb{F}_{\text{bd}}(\mathbb{R}^n)$ is a dense subset of $(\mathbb{F}^p(\mathbb{R}^n), \rho_p)$ and $(\mathbb{F}_{\text{bd}}(\mathbb{R}^n), \rho_p)$ is separable, $(\mathbb{F}^p(\mathbb{R}^n), \rho)$ is also separable. Therefore, it remains to prove the completeness of $(\mathbb{F}^p(\mathbb{R}^n), \rho)$.

Let $\{\tilde{U}_m \in \mathbb{F}^p(\mathbb{R}^n); m = 1, 2, \dots\}$ be some Cauchy sequence and let $j_{\mathbb{F}^p(\mathbb{R}^n)}$ be the embedding of $\mathbb{F}^p(\mathbb{R}^n)$ into $\mathbb{L}^p([0, 1] \times S^{n-1})$ given by (5.7). Then, $\{j_{\mathbb{F}^p(\mathbb{R}^n)}(\tilde{U}_m); m = 1, 2, \dots\}$ is a Cauchy sequence with respect to the norm $\|\cdot\|_p$ defined by

$$\|j_{\mathbb{F}^p(\mathbb{R}^n)}(\tilde{U}) - j_{\mathbb{F}^p(\mathbb{R}^n)}(\tilde{V})\|_p = \rho_p(\tilde{U}, \tilde{V})$$

for arbitrary $\tilde{U}, \tilde{V} \in \mathbb{F}^p(\mathbb{R}^n)$. Since $(\mathbb{L}^p([0, 1] \times S^{n-1}), \|\cdot\|_p)$ is a Banach space, there exists some $f \in \mathbb{L}^p([0, 1] \times S^{n-1})$ with

$$\lim_{m \rightarrow \infty} \|j_{\mathbb{F}^p(\mathbb{R}^n)}(\tilde{U}_m) - f\|_p = 0.$$

Applying Fubini's theorem, we can find some $\mu_{[0, 1]}$ -null set N and a subsequence $\{\tilde{U}_{\psi(m)}; m = 1, 2, \dots\}$ with

$$\lim_{m \rightarrow \infty} \int_{S^{n-1}} |\Phi_{\tilde{U}_{\psi(m)}}(\alpha, x) - f(\alpha, x)|^p d\mu_{S^{n-1}}(x) = 0$$

for $\alpha \in I \setminus N$. Using Minkowski's inequality, we have

$$\begin{aligned} & \left(\int |\Phi_{\tilde{U}_{\psi(m)}}(\alpha, x) - f(\alpha, x)|^p d\mu_{S^{n-1}}(x) \right)^{\frac{1}{p}} \\ & \geq \left| \left(\int_{S^{n-1}} |\Phi_{\tilde{U}_{\psi(m)}}(\alpha, x) - \Phi_{\tilde{U}_{\psi(\ell)}}(\alpha, x)|^p d\mu_{S^{n-1}}(x) \right)^{\frac{1}{p}} \right. \\ & \quad \left. - \left(\int_{S^{n-1}} |(f(\alpha, x) - \Phi_{\tilde{U}_{\psi(\ell)}}(\alpha, x)|^p d\mu_{S^{n-1}}(x) \right)^{\frac{1}{p}} \right|. \end{aligned}$$

Hence, setting $m, \ell \rightarrow \infty$, it follows

$$\begin{aligned} & \left(\int_{S^{n-1}} |\text{sp}(\overline{\text{co}}[\tilde{U}_{\psi(m)}]\alpha - \text{sp}(\overline{\text{co}}[\tilde{U}_{\psi(\ell)}]\alpha)|^p d\mu_{S^{n-1}}(x) \right)^{\frac{1}{p}} \\ & = \delta_p(\overline{\text{co}}[\tilde{U}_{\psi(m)}], \overline{\text{co}}[\tilde{U}_{\psi(\ell)}]) \rightarrow 0, \end{aligned}$$

which means that $\{\overline{\text{co}}[\tilde{U}_{\psi(m)}]; m = 1, 2, \dots\}$ is a Cauchy sequence with respect to the metric δ_p for $\alpha \in I \setminus N$. Hence, due to the completeness of the space of non-void compact convex subsets of $\mathbb{R}^n$ with the metric δ_p , there exists a non-void compact convex subset K_α for every $\alpha \in I \setminus N$ with

$$\lim_{m \rightarrow \infty} \delta_p(\overline{\text{co}}[\tilde{U}_{\psi(m)}], K_\alpha) = 0. \quad (5.8)$$

Drawing on Theorem II-2 in [21], it can be shown that

$$K_\alpha = \bigcap_{m=1}^{\infty} \text{cl.} \left(\bigcup_{\ell \geq m} \overline{\text{co}}[\tilde{U}_{\psi(\ell)}]\alpha \right) \subseteq \bigcap_{m=1}^{\infty} \text{cl.} \left(\bigcup_{\ell \geq m} \overline{\text{co}}[\tilde{U}_{\psi(\ell)}]\beta \right) = K_\beta$$

for $0 < \beta < \alpha \leq 1$ and $\alpha, \beta \in I \setminus N$. We can define a system of bounded subsets $\{\overline{\text{co}}[\tilde{U}]\alpha | \alpha \in I\}$ by

$$\overline{\text{co}}[\tilde{U}]\alpha = \bigcap_{\beta \in (0, \alpha) \setminus N} K_\beta,$$

satisfying $\overline{\text{co}}[\tilde{U}]\alpha \subseteq \overline{\text{co}}[\tilde{U}]\beta$ for $0 < \beta < \alpha \leq 1$. For a fixed $x \in \mathbb{Q}^n \cap S^{n-1}$, the real-valued mapping ϕ_x on I defined by

$$\phi_x(\alpha) = \begin{cases} \text{sp}(x, K_\alpha) & \text{for } \alpha \in I \setminus N \\ \text{sp}(x, \overline{\text{co}}[\tilde{U}]\alpha) & \text{for } \alpha \in N \end{cases}$$

is increasing. Hence, there exists some at most countable set $N_x \subseteq I$ such that the restriction $\phi_x|_{I \setminus N_x}$ of ϕ_x to $I \setminus N_x$ is continuous. Then, defining

$$\bar{N} = N \bigcup_{x \in \mathbb{Q}^n \cap S^{n-1}} N_x,$$

every mapping $\phi_x|_{I \setminus \bar{N}}$ for $x \in \mathbb{Q}^n \cap S^{n-1}$ is continuous.

Let $\alpha \in I \setminus \bar{N}$ and $\varepsilon > 0$. Since the support function of a compact convex subset of $\mathbb{R}^n$ is continuous, we can find

$$\max \left\{ |\text{sp}(x, K_\alpha) - \text{sp}(y, K_\alpha)|, |\text{sp}(x, \overline{\text{co}}[\tilde{U}]_\alpha) - \text{sp}(y, \overline{\text{co}}[\tilde{U}]_\alpha)| \right\} < \frac{\varepsilon}{2}.$$

Moreover, there is an monotone increasing sequence $\{\alpha_m \in I \setminus \bar{N}; m = 1, 2, \dots\}$ converging to α . According to

$$K_\alpha \subseteq \overline{\text{co}}[\tilde{U}]_\alpha \subseteq K_{\alpha_n}$$

for $n = 1, 2, \dots$, it follows that

$$\phi_x(\alpha_n) = \text{sp}(x, K_{\alpha_n}) \geq \text{sp}(x, \overline{\text{co}}[\tilde{U}]_\alpha) \geq \text{sp}(x, K_\alpha) = \phi_x(\alpha).$$

Therefore, $\text{sp}(x, K_\alpha) = \text{sp}(x, \overline{\text{co}}[\tilde{U}]_\alpha)$ due to the continuity of $\phi_x|_{I \setminus \bar{N}}$, and

$$|\text{sp}(x, K_\alpha) - \text{sp}(x, \overline{\text{co}}[\tilde{U}]_\alpha)| \leq |\text{sp}(x, K_\alpha) - \text{sp}(y, K_\alpha)| + |\text{sp}(y, K_\alpha) - \text{sp}(y, \overline{\text{co}}[\tilde{U}]_\alpha)| \leq \varepsilon.$$

Hence, we have

$$\overline{\text{co}}[\tilde{U}]_\alpha = K_\alpha \quad \text{and} \quad \delta_p(K_\alpha, \overline{\text{co}}[\tilde{U}]_\alpha) = 0 \quad (5.9)$$

for $\alpha \in I \setminus \bar{N}$. Then

$$\overline{\text{co}}[\tilde{U}]_\alpha = \bigcap_{\beta \in (0, \alpha)} \overline{\text{co}}[\tilde{U}]_\beta$$

follows immediately since $\alpha \in I \setminus \bar{N}$ is a dense subset of $[0, 1]$, and using Proposition 2.1, it can be shown that $\{\overline{\text{co}}[\tilde{U}]_\alpha | \alpha \in I\}$ is left-continuous, which means it satisfies the condition (ii) of Definition 4.1. Then, we can consider that $[\tilde{U}] = \{[\tilde{U}]_\alpha | \alpha \in I\}$ is a set representation of a fuzzy set $\tilde{U} \in \mathbb{F}(\mathbb{R}^n)$. Finally, in view of (5.8), (5.9) and using Minkowski's inequality, we can conclude that

$$\begin{aligned} \rho_p(\tilde{U}_m, \tilde{U}) &= \left(\int |\varphi_{\tilde{U}}(\alpha, x) - f(\alpha, x)|^p d\mu_{S^{n-1}}(x) \right)^{\frac{1}{p}} \\ &\leq \left(\int_0^1 \delta_p(\overline{\text{co}}[\tilde{U}_m]_\alpha, K_\alpha)^p d\mu_{[0,1]}(\alpha) \right)^{\frac{1}{p}} \\ &\quad + \left(\int_0^1 \delta_p(K_\alpha, \overline{\text{co}}[\tilde{U}]_\alpha)^p d\mu_{[0,1]}(\alpha) \right)^{\frac{1}{p}} \\ &\rightarrow 0 \quad \text{as } m \rightarrow \infty. \end{aligned}$$

□

The Steiner centroid of a nonempty compact convex set is also generalized to the fuzzy sets in $\mathbb{F}^2(\mathbb{R}^n)$. It is defined as follows:

$$\sigma(\tilde{U}) = n \cdot \int_0^1 \int_{S^{n-1}} x \cdot \varphi_{\tilde{U}}(\alpha, x) d\mu_{[0,1]}(\alpha) d\mu_{S^{n-1}}(x).$$

The Steiner centroid for fuzzy set $\tilde{U} \in \mathbb{F}^2(\mathbb{R}^n)$, $\sigma(\tilde{U})$ satisfies $\sigma(-\tilde{U}) = -\sigma(\tilde{U})$. The Steiner centroid for fuzzy sets in $\mathbb{F}^2(\mathbb{R}^n)$ preserve the linearity

$$\sigma(a\tilde{U} + b\tilde{V}) = a \cdot \sigma(\tilde{U}) + b \cdot \sigma(\tilde{V})$$

for $a, b \in \mathbb{R}$, $\tilde{U}, \tilde{V} \in \mathbb{F}^2(\mathbb{R}^n)$, and $\tilde{U}^0 = \tilde{U} - \sigma(\tilde{U})$ is centered so that $\sigma(\tilde{U}^0) = 0$ holds.

The space $(\mathbb{F}^2(\mathbb{R}^n), \rho_2)$ is isomorphic to a closed convex cone in the Hilbert space $\mathbb{L}_2([0, 1] \times S^{n-1})$ equipped with the inner product

$$(\tilde{U}, \tilde{V}) = \int_0^1 \int_{S^{n-1}} \varphi_{\tilde{U}}(\alpha, p) \cdot \varphi_{\tilde{V}}(\alpha, p) d\mu_{[0,1]}(\alpha) d\mu_{S^{n-1}}(x).$$

Furthermore, for the metric ρ_2 we have the following properties:

- (i) The Steiner centroid is a characteristic point of a fuzzy set in the sense of

$$\inf \left\{ \rho_2(\{a\}, \tilde{U}) \mid a \in \mathbb{R}^n \right\} = \rho_2(\{\sigma(\tilde{U})\}, \tilde{U}).$$

- (ii) Furthermore

$$\rho_2(\tilde{U}, \tilde{V}) = \rho_2(\tilde{U}^0, \tilde{V}^0) + \|\sigma(\tilde{U}) - \sigma(\tilde{V})\|^2,$$

where $\sigma(\tilde{U})$ is the Steiner centroid of $\tilde{U}$ and $\tilde{U}^0 = \tilde{U} - \sigma(\tilde{U})$ (analogous $\tilde{V}^0 = \tilde{V} - \sigma(\tilde{V})$).

6 Conclusions

In this paper, the author has investigated a metric space for a class of fuzzy sets, where fuzzy sets are represented by the general set representation approach.

First, maintaining the viewpoint of the mult-valued log, a class of fuzzy sets has been defined based on the general set representation approach. Then, a metric on the space of fuzzy sets has been introduced, where the distance between two fuzzy sets was defined by using the support functions of fuzzy sets. Furthermore, it has been proved that the proposed metric space is complete, separable one, which will be very useful properties to study fuzzy random vectors proposed by the author[5, 6, 7].

It should be noted that all the results in this paper are inspired by the results given by Krätschmer[1, 2].

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