

# On Basic Arithmetic Operations for Fuzzy Sets Obtained as Vague Perceptions of Crisp Phenomena

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## Abstract

In this paper, some of basic properties for a class of fuzzy sets in a metric space are investigated intensively with the help of set representations of fuzzy sets.

## 1 Introduction

The purpose of this paper is to study the properties of fuzzy sets, which will become basic tools for investigating fuzzy random vectors given as a model for vague perceptions of random phenomena [1~7].

Section 2 is devoted to investigate the basic properties of fuzzy sets in a metric space, where a fuzzy set is defined through the set representation approach, i.e., given by a triple of the basic space, the set representation and the predicate. The set representation approach adopted in this section is utilized throughout following

sections. In Section 3, the functions of fuzzy sets and the operations between fuzzy sets are studied with the help of the extension principle concept [8]. Finally, some of basic properties of arithmetic means and variances for fuzzy sets are investigated in Section 4.

It should be noted that all the results in this paper are extended or modified versions of [9].

## 2 Set Representation Approach of Fuzzy Sets

In this paper, a fuzzy set  $\tilde{U}$  given as a vague perception of  $u_o \in X$  is defined by the triple

$$\tilde{U} = (X, \{\tilde{U}_\alpha \mid \alpha \in (0, 1)\}, s_{\tilde{U}}), \quad (2.1)$$

where  $X$  is called the basic space;  $s_{\tilde{U}}$  is the predicate, i. e.,  $s_{\tilde{U}}: X \rightarrow \mathcal{P}$  with  $\mathcal{P}$  the "universe of discourse" defined by a set of statements, assigns a statement

$$\begin{aligned} s_{\tilde{U}}(x) &= \{x \text{ coincides with } u_o\} \\ &= \{x = u_o\} \end{aligned} \quad (2.2)$$

to each element  $x \in X$ ; and  $\{\tilde{U}_\alpha \mid \alpha \in (0, 1)\}$  is the family of subsets of  $X$  satisfying

$$L_\alpha \tilde{U} \subseteq \tilde{U}_\alpha \subseteq L_{\bar{\alpha}} \tilde{U} \quad \text{for } \forall \alpha \in (0, 1) \quad (2.3)$$

In (2.3),  $L_\alpha \tilde{U}$  and  $L_{\bar{\alpha}} \tilde{U}$  are the strong cut and the level set of  $\tilde{U}$  at the level  $\alpha$  defined respectively by

$$L_\alpha \tilde{U} = \{x \mid (x \in X) \wedge ((\tilde{U})(x) > \alpha)\} \quad \text{for } \alpha \in [0, 1) \quad (2.4a)$$

$$L_{\bar{\alpha}} \tilde{U} = \{x \mid (x \in X) \wedge ((\tilde{U})(x) \geq \alpha)\} \quad \text{for } \alpha \in (0, 1], \quad (2.4b)$$

where  $(\tilde{U})(x)$  is the membership function of  $\tilde{U}$  given by

$$(\tilde{U})(x) = t(s_{\tilde{U}}(x)) \quad (2.5)$$

and  $t(*)$  in (2.5) is the truth function of  $*$  in the sense of multi-valued logic. The crisp point  $u_0$  in  $X$ , which gives the vague perception, i.e., the fuzzy set  $\tilde{U}$ , is called the original point of  $\tilde{U}$ . The definition of fuzzy sets adopted in this paper is a modified one originally proposed by Kwakernaak [10]. Furthermore, we call the family of subsets  $\{\tilde{U}_\alpha \mid \alpha \in (0, 1)\}$  defined by (2.3) as the set representation of a fuzzy set  $\tilde{U}$ . This view is slightly but not essentially different from its usual definition (see e.g. [11] or [12]).

Since

$$\min(\alpha, I_{L_{\tilde{U}}\tilde{U}}(x)) = \begin{cases} \alpha, & \text{if } (\tilde{U})(x) \geq \alpha \\ 0, & \text{if } (\tilde{U})(x) < \alpha, \end{cases} \quad (2.6)$$

it follows

$$\begin{aligned} (\tilde{U})(x) &= \sup\{\alpha \mid (\tilde{U})(x) \geq \alpha\} \\ &= \sup\{\min(\alpha, I_{L_{\tilde{U}}\tilde{U}}(x))\}, \end{aligned} \quad (2.7)$$

Hence, if  $(\tilde{U})(x) > 0$ , we can infer

$$\begin{aligned} (\tilde{U})(x) &= \sup\{\min(\alpha, I_{L_{\tilde{U}}\tilde{U}}(x))\} \\ &= \sup\{\alpha \mid (\alpha \in [0, 1]) \wedge (x \in L_{\tilde{U}}\tilde{U})\} \\ &= \sup\{\alpha \mid (\alpha \in (0, 1)) \wedge (x \in L_{\tilde{U}}\tilde{U})\} \\ &\geq \sup\{\alpha \mid (\alpha \in (0, 1)) \wedge (x \in \tilde{U}_\alpha)\} \\ &\geq \sup\{\alpha \mid (\alpha \in (0, 1)) \wedge (x \in L_\alpha \tilde{U})\} \\ &= (\tilde{U})(x). \end{aligned} \quad (2.8)$$

Therefore, defining  $\sup \emptyset = 0$ , we can conclude

$$(\tilde{U})(x) = \sup\{\alpha \mid (\alpha \in (0, 1)) \wedge (x \in \tilde{U}_\alpha)\}$$

and the following theorem is obtained :

**Theorem 2. 1** *Let  $\tilde{U} = (X, \{\tilde{U}_\alpha \mid \alpha \in (0, 1)\}, s_{\tilde{U}})$  be an arbitrary fuzzy set. Then, we have*

$$(\tilde{U})(x) = \sup\{\alpha \mid (\alpha \in (0, 1)) \wedge (x \in \tilde{U}_\alpha)\} \quad (2. 9)$$

with  $\sup \emptyset = 0$ .

As seen from the definition of fuzzy sets given by (2. 1), there may be many fuzzy sets which have same strong cuts and level sets at every level  $\alpha \in (0, 1)$ . This causes the following definition of the equivalence of fuzzy sets :

**Definition 2. 1** *Two fuzzy sets  $\tilde{U}$  and  $\tilde{V}$  is called equivalent and denoted by  $\tilde{U} = \tilde{V}$ , if*

$$L_\alpha \tilde{U} = L_\alpha \tilde{V} \quad \text{and} \quad L_{\bar{\alpha}} \tilde{U} = L_{\bar{\alpha}} \tilde{V} \quad \text{for} \quad \forall \alpha \in (0, 1). \quad (2. 10)$$

It is easy to see from Theorem 2. 1 that the system  $\{\tilde{V}_\alpha \mid \alpha \in (0, 1)\}$  with the relation

$$L_\alpha \tilde{U} \subseteq \tilde{V}_\alpha \subseteq L_{\bar{\alpha}} \tilde{U} \quad \text{for} \quad \forall \alpha \in (0, 1) \quad (2. 11)$$

satisfies

$$\begin{aligned} (\tilde{U})(x) &= \sup\{\alpha \mid (\alpha \in (0, 1)) \wedge (x \in \tilde{U}_\alpha)\} \\ &= \sup\{\alpha \mid (\alpha \in (0, 1)) \wedge (x \in \tilde{V}_\alpha)\} \\ &= (\tilde{V})(x), \end{aligned} \quad (2. 12)$$

and hence  $\tilde{U}$  and  $\tilde{V}$  satisfy (2.10). Therefore, the following corollary holds:

**Corollary 2.1** *If the following relation*

$$L_\alpha \tilde{U} \subseteq \tilde{V}_\alpha \subseteq L_{\bar{\alpha}} \tilde{U} \quad \text{for } \forall \alpha \in (0, 1) \quad (2.13)$$

*holds, then  $\tilde{U} = \tilde{V}$ , where  $\{\tilde{V}_\alpha \mid \alpha \in (0, 1)\}$  is the set representation of  $\tilde{V}$ .*

**Theorem 2.2** *Let  $\tilde{U} = (X, \{\tilde{U}_\alpha \mid \alpha \in (0, 1)\}, s_{\tilde{U}})$  be an arbitrary fuzzy set. Then, we have*

$$L_\alpha \tilde{U} = \bigcup_{\beta \in (\alpha, 1)} \tilde{U}_\beta \quad \text{for } \alpha \in [0, 1) \quad (2.14a)$$

and

$$L_{\bar{\alpha}} \tilde{U} = \bigcap_{\beta \in (0, \alpha)} \tilde{U}_\beta \quad \text{for } \alpha \in (0, 1]. \quad (2.14b)$$

**Note:** For the derivation of (2.14a) and (2.14b), it should remind of

$$\begin{aligned} \bigcup_{\beta \in (\alpha, 1)} \tilde{U}_\beta &= \{x \mid \exists \beta \in (\alpha, 1) (x \in \tilde{U}_\beta)\} \\ &= \{x \mid \text{there exists } \beta \in (\alpha, 1) \text{ such that } x \in \tilde{U}_\beta\} \end{aligned} \quad (2.15a)$$

and

$$\begin{aligned} \bigcap_{\beta \in (0, \alpha)} \tilde{U}_\beta &= \{x \mid \forall \beta \in (0, \alpha) (x \in \tilde{U}_\beta)\} \\ &= \{x \mid \text{for any } \beta \in (0, \alpha), x \in \tilde{U}_\beta\} \end{aligned} \quad (2.15b)$$

holds.

Hereafter, let the basic space  $X$  be a metric space and let  $\{\tilde{U}_\alpha \mid \alpha \in (0, 1)\}$  be the set representation of the fuzzy set  $\tilde{U}$ . Then, the collection of all fuzzy sets in  $X$  is denoted by  $\mathcal{G}(X)$  when it satisfies that

- (i)  $\bigcap_{\alpha \in (0, 1)} \tilde{U}_\alpha \neq \emptyset$
- (ii)  $\tilde{U}_\alpha$  is compact for each  $\alpha \in (0, 1)$
- (iii)  $\text{supp. } \tilde{U}$  defined by  $\text{supp. } \tilde{U} = \text{cl. } \bigcup_{\alpha \in (0, 1)} \tilde{U}_\alpha$  is also compact.

It is clear from (2. 10) and (2. 14a) that if  $\tilde{U}$  is equivalent to  $\tilde{V}$ , i. e.,  $\tilde{U} = \tilde{V}$ , then  $\text{supp. } \tilde{U} = \text{supp. } \tilde{V}$ . In the following, we do not discriminate equivalent fuzzy sets. The family of fuzzy sets is denoted by  $\mathcal{G}_c(X)$  when its each element  $\tilde{U}$  is the member of  $\mathcal{G}(X)$  and its set representation is composed of convex sets. The fuzzy sets on the basic space  $X$  will be represented by using capital letters with tilde.

**Theorem 2. 3** *Let  $\tilde{U}$  be a fuzzy set in  $\mathcal{G}(X)$ . Then,  $L_{\bar{\alpha}}\tilde{U}$  is compact each  $\alpha \in (0, 1)$ . Furthermore, if  $\tilde{U} \in \mathcal{G}_c(X)$ ,  $L_{\bar{\alpha}}\tilde{U}$ ,  $L_{\alpha}\tilde{U}$  for  $\forall \alpha \in (0, 1)$ , and  $\text{supp. } \tilde{U}$  are convex.*

The proofs of Theorems 2. 2 and 2. 3 are given in Appendix A.

### 3 Operations on Fuzzy Sets

#### 3. 1 Fuzzy Functions

Let  $f: X \rightarrow Y$  be a mapping from a metric space  $X$  to another metric

space  $Y$ . Then, the image of  $A \subset X$  is the subset of  $Y$  defined by

$$f(A) = \{y \in Y \mid y = f(x) \text{ with } x \in A\}. \quad (3.1)$$

Suppose that  $\tilde{U} = (X, \{\tilde{U}_\alpha \mid \alpha \in (0, 1)\}, s_{\tilde{U}})$  is a fuzzy set, and consider the statement

$$s_{f(\tilde{U})}(y) = \{y \text{ coincides with } f(u_o)\}, \quad (3.2)$$

where  $u_o$  is the original point of  $\tilde{U}$ . Then, the above statement should be rewritten by the following composite statement:

$$\begin{aligned} s_{f(\tilde{U})}(y) &= \bigvee_{x \in X} \{(y = f(x)) \wedge (x = u_o)\} \\ &= \bigvee_{x \in X} \{(y = f(x)) \wedge (s_{\tilde{U}}(x))\}, \end{aligned} \quad (3.3)$$

where  $s_{\tilde{U}}(x)$  is the statement associated with the predicate of  $\tilde{U}$ . Therefore, according to the concept of the extension principle of fuzzy set proposed by Zadeh [8], it may be reasonable to evaluate the truth value of  $s_{f(\tilde{U})}(y)$  as follows:

$$t(s_{f(\tilde{U})}(y)) = \sup\{(\tilde{U})(x) \mid y = f(x) \text{ with } x \in X\}. \quad (3.4)$$

Let assume

$$\eta \in \{y \mid t(s_{f(\tilde{U})}(y)) > \alpha\}. \quad (3.5)$$

Then, using (3.4) we can find a  $\zeta \in X$  such that  $\eta = f(\zeta)$  with  $(\tilde{U})(\zeta) > \alpha$  and this follows

$$\eta \in \{y \mid y = f(x) \text{ with } x \in L_\alpha \tilde{U}\}, \quad (3.6)$$

or equivalently

$$\{y \mid t(s_{f(\tilde{U})}(y)) > \alpha\} \subseteq f(L_\alpha \tilde{U}). \quad (3.7)$$

Furthermore, if

$$\eta \in f(L_{\bar{\alpha}} \tilde{U}) = \{y \mid y = f(x) \text{ with } x \in L_{\bar{\alpha}} \tilde{U}\}, \quad (3.8)$$

there exists  $\zeta \in X$  such that  $\eta = f(\zeta)$  with  $\zeta \in L_{\bar{\alpha}} \tilde{U}$ , which implies

$$\begin{aligned} \eta &\in \{y \mid \sup\{(\tilde{U})(x) \mid y = f(x) \text{ with } x \in X\} \geq \alpha\} \\ &= \{y \mid t(s_{f(\tilde{U})}(y)) \geq \alpha\} \end{aligned} \quad (3.9)$$

or equivalently

$$f(L_{\bar{\alpha}} \tilde{U}) \subseteq \{y \mid t(s_{f(\tilde{U})}(y)) \geq \alpha\}, \quad (3.10)$$

Invoking here the relation

$$L_\alpha \tilde{U} \subseteq \tilde{U}_\alpha \subseteq L_{\bar{\alpha}} \tilde{U} \text{ for } \forall \alpha \in (0, 1)$$

given by (2.3), and using (3.7) and (3.10), we have

$$\begin{aligned} \{y \mid t(s_{f(\tilde{U})}(y)) > \alpha\} &\subseteq f(L_\alpha \tilde{U}) \subseteq f(\tilde{U}_\alpha) \\ &\subseteq f(L_{\bar{\alpha}} \tilde{U}) \subseteq \{y \mid t(s_{f(\tilde{U})}(y)) \geq \alpha\} \end{aligned} \quad (3.11)$$

for  $\forall \alpha \in (0, 1)$ . Therefore, the following definition concerned with the fuzzy function should be reasonable:

**Definition 3.1** The fuzzy function  $\tilde{V} = f(\tilde{U})$  as the generalization of the mapping of  $f: X \rightarrow Y$  is given by

$$\tilde{V} = f(\tilde{U}) = (Y, \{f(\tilde{U}_\alpha) \mid \alpha \in (0, 1)\}, s_{f(\tilde{U})}). \quad (3.12)$$

It should be clear that the strong cut and level set of  $\tilde{V}$  are given respectively by

$$L_\alpha \tilde{V} = L_\alpha f(\tilde{U}) = \{y \mid (f(\tilde{U}))(y) > \alpha\} \quad (3.13a)$$

$$L_{\bar{\alpha}} \tilde{V} = L_{\bar{\alpha}} f(\tilde{U}) = \{y \mid (f(\tilde{U}))(y) \geq \alpha\} \quad (3.13b)$$

with

$$(f(\tilde{U}))(y) = t(s_{f(\tilde{U})}(y)). \quad (3.13c)$$

Hence, from (3.11), (3.13a) and (3.13b), it is clear that

$$L_\alpha f(\tilde{U}) \subseteq f(L_\alpha \tilde{U}) \subseteq f(\tilde{U}_\alpha) \subseteq f(L_{\bar{\alpha}} \tilde{U}) \subseteq L_{\bar{\alpha}} f(\tilde{U}). \quad (3.14)$$

Furthermore, assume that

$$\begin{aligned} \eta \in f(L_\alpha \tilde{U}) &= \{y \mid y = f(x) \text{ with } x \in L_\alpha \tilde{U}\} \\ &= \{y \mid (y = f(x)) \wedge ((\tilde{U})(x) > \alpha)\}. \end{aligned} \quad (3.15)$$

Then, we can find that there is at least one element  $\zeta \in L_\alpha \tilde{U}$  satisfying  $\eta = f(\zeta)$ , which yields

$$\begin{aligned} \eta = f(\zeta) &\in \{y \mid (y = f(x)) \wedge ((\tilde{U})(x) > \alpha)\} \\ &\subseteq \{y \mid \sup\{(\tilde{U})(x) \mid y = f(x) \text{ with } x \in X\} > \alpha\} \\ &= \{y \mid t(s_{f(\tilde{U})}(y)) > \alpha\} \\ &= L_\alpha f(\tilde{U}), \end{aligned} \quad (3.16)$$

or equivalently

$$f(L_\alpha \tilde{U}) \subseteq L_\alpha f(\tilde{U}) \quad \text{for } \alpha \in [0, 1). \quad (3.17)$$

The inclusion of  $L_\alpha f(\tilde{U})$  to  $f(L_\alpha \tilde{U})$  for  $\alpha \in (0, 1)$  has already been shown in (3.14), and that for  $\alpha = 0$  is also obvious from the

derivation process of (3. 14). Therefore, the following proposition holds :

**Proposition 3. 1**

$$L_\alpha f(\tilde{U}) = f(L_\alpha \tilde{U}) \quad \text{for } \alpha \in [0, 1]. \quad (3. 18)$$

For the level sets of  $\tilde{U}$ , the following theorem is obtained :

**Theorem 3. 1** *Let  $\tilde{U} = (X, \{\tilde{U}_\alpha \mid \alpha \in (0, 1)\}, s_{\tilde{U}})$  be a fuzzy set in  $\mathcal{G}(X)$ , and  $f: X \rightarrow Y$  be a continuous mapping from a metric space  $X$  to another metric space  $Y$ . Then,*

$$f(\tilde{U}) = (Y, \{f(\tilde{U}_\alpha) \mid \alpha \in (0, 1)\}, s_{f(\tilde{U})}) \quad (3. 19)$$

*is the fuzzy set in  $\mathcal{G}(Y)$  satisfying*

$$L_\alpha f(\tilde{U}) = f(L_\alpha \tilde{U}) \quad \text{for } \forall \alpha \in (0, 1]. \quad (3. 20)$$

### 3. 2 Operations Between Fuzzy Sets

For the arithmetic operations between fuzzy sets, consider the following statements :

$$s_{f(\tilde{U}_1, \dots, \tilde{U}_n)}(y) = \{y \text{ coincides with } f(u_{1,o}, \dots, u_{n,o})\}, \quad (3. 21)$$

where  $u_{i,o}$  is the original point of  $\tilde{U}_i$  and  $f: X_1 \times \dots \times X_n \rightarrow Y$  be a mapping from the product space of  $X_i (i=1, \dots, n)$  to  $Y$ . Then, the above statement should be rewritten as the following composite statement :

$$\begin{aligned}
s_{f(\tilde{v}_1, \dots, \tilde{v}_n)}(y) &= \bigvee_{x \in X_1 \times \dots \times X_n} \left\{ (y = f(x_1, \dots, x_n)) \wedge \left( \bigwedge_{i=1}^n (x_i = u_i) \right) \right\} \\
&= \bigvee_{x \in X_1 \times \dots \times X_n} \left\{ (y = f(x_1, \dots, x_n)) \wedge \left( \bigwedge_{i=1}^n (s_{\tilde{v}_i}(x_i)) \right) \right\}, \quad (3.22)
\end{aligned}$$

where  $s_{\tilde{v}_i}(x)$  is the statement associated with the predicate of  $\tilde{U}_i$ . Therefore, according to the concept of the extension principle of fuzzy set, it may be reasonable to evaluate the truth value of  $s_{f(\tilde{v}_1, \dots, \tilde{v}_n)}(y)$  as follows:

$$\begin{aligned}
t(s_{f(\tilde{v}_1, \dots, \tilde{v}_n)}(y)) &= \sup \left\{ \min_{i \in \{1, \dots, n\}} (\tilde{U}_i)(x_i) \mid (y = f(x_1, \dots, x_n)) \right. \\
&\quad \left. \wedge \left( \bigwedge_{i=1}^n (x_i \in X_i) \right) \right\}. \quad (3.23)
\end{aligned}$$

Let assume

$$\eta \in \{y \mid t(s_{f(\tilde{v}_1, \dots, \tilde{v}_n)}(y)) > \alpha\}. \quad (3.24)$$

Then, using (3.23) we can find a  $\zeta_i \in X_i (i=1, \dots, n)$  such that  $\eta = f(\zeta_1, \dots, \zeta_n)$  with  $\min_{i \in \{1, \dots, n\}} (\tilde{U}_i)(\zeta_i) > \alpha$  and this follows

$$\eta \in \left\{ y \mid (y = f(x_1, \dots, x_n)) \wedge \left( \bigwedge_{i=1}^n (x_i \in L_\alpha \tilde{U}_i) \right) \right\}, \quad (3.25)$$

or equivalently

$$\{y \mid t(s_{f(\tilde{v}_1, \dots, \tilde{v}_n)}(y)) > \alpha\} \subseteq f(L_\alpha \tilde{U}_1, \dots, L_\alpha \tilde{U}_n). \quad (3.26)$$

Furthermore, if

$$\eta \in f(L_{\bar{\alpha}} \tilde{U}_1, \dots, L_{\bar{\alpha}} \tilde{U}_n)$$

$$= \{y \mid (y = f(x_1, \dots, x_n)) \wedge \left( \bigwedge_{i=1}^n (x_i \in L_{\bar{\alpha}} \tilde{U}_i) \right)\}, \quad (3.27)$$

there exists  $\zeta_i \in X_i (i=1, \dots, n)$  such that  $\eta = f(\zeta_1, \dots, \zeta_n)$  with  $\zeta_i \in L_{\bar{\alpha}} \tilde{U}_i$ , which implies

$$\begin{aligned} \eta \in \{y \mid \sup \{ \min_{i \in \{1, \dots, n\}} (\tilde{U}_i)(x) \mid (y = f(x_1, \dots, x_n)) \wedge \left( \bigwedge_{i=1}^n (x_i \in X_i) \right) \} \geq \alpha \} \\ = \{y \mid t(s_{f(\tilde{U}_1, \dots, \tilde{U}_n)}(y)) \geq \alpha \} \end{aligned} \quad (3.28)$$

or equivalently

$$f(L_{\bar{\alpha}} \tilde{U}_1, \dots, L_{\bar{\alpha}} \tilde{U}_n) \subseteq \{y \mid t(s_{f(\tilde{U}_1, \dots, \tilde{U}_n)}(y)) \geq \alpha\}. \quad (3.29)$$

Then, applying (2.3) to (3.26) and (3.29), we have

$$\begin{aligned} \{y \mid t(s_{f(\tilde{U}_1, \dots, \tilde{U}_n)}(y)) > \alpha\} \subseteq f(L_{\alpha} \tilde{U}_1, \dots, L_{\alpha} \tilde{U}_n) \subseteq f(\tilde{U}_{1,\alpha}, \dots, \tilde{U}_{n,\alpha}) \\ \subseteq f(L_{\bar{\alpha}} \tilde{U}_1, \dots, L_{\bar{\alpha}} \tilde{U}_n) \subseteq \{y \mid t(s_{f(\tilde{U}_1, \dots, \tilde{U}_n)}(y)) \geq \alpha\} \end{aligned} \quad (3.30)$$

for  $\forall \alpha \in (0, 1)$ . Therefore, the following definition concerned with operations between fuzzy sets should be reasonable:

**Definition 3.2** The fuzzy function  $\tilde{V} = f(\tilde{U}_1, \dots, \tilde{U}_n)$  as the generalization of the mapping  $f: X_1 \times \dots \times X_n \rightarrow Y$  is given by

$$\tilde{V} = f(\tilde{U}_1, \dots, \tilde{U}_n) = (Y, \{f(\tilde{U}_{1,\alpha}, \dots, \tilde{U}_{n,\alpha}) \mid \alpha \in (0, 1)\}, s_{f(\tilde{U}_1, \dots, \tilde{U}_n)}), \quad (3.31)$$

where  $s_{f(\tilde{U}_1, \dots, \tilde{U}_n)}$  is the predicate associated with the statement given by (3.21); and  $\tilde{U}_{i,\alpha}$  is the element of the set representation of  $\tilde{U}_i$  for each  $i = 1, \dots, n$ .

It should be clear that the strong cut and level set of  $\tilde{V}$  are

given by

$$L_\alpha \tilde{V} = L_\alpha f(\tilde{U}_1, \dots, \tilde{U}_n) = \{y \mid (f(\tilde{U}_1, \dots, \tilde{U}_n))(y) > \alpha\} \quad (3.32a)$$

$$L_{\bar{\alpha}} \tilde{V} = L_{\bar{\alpha}} f(\tilde{U}_1, \dots, \tilde{U}_n) = \{y \mid ((f(\tilde{U}_1, \dots, \tilde{U}_n))(y) \geq \alpha)\} \quad (3.32b)$$

with

$$(f(\tilde{U}_1, \dots, \tilde{U}_n))(y) = t(s_{f(\tilde{v}_1, \dots, \tilde{v}_n)}(y)). \quad (3.32c)$$

Hence, from (3.30), (3.32a) and (3.32b), it is clear that

$$\begin{aligned} L_\alpha f(\tilde{U}_1, \dots, \tilde{U}_n) &\subseteq f(L_\alpha \tilde{U}_1, \dots, L_\alpha \tilde{U}_n) \subseteq f(\tilde{U}_{1,\alpha}, \dots, \tilde{U}_{n,\alpha}) \\ &\subseteq f(L_{\bar{\alpha}} \tilde{U}_1, \dots, L_{\bar{\alpha}} \tilde{U}_n) \subseteq L_{\bar{\alpha}} f(\tilde{U}_1, \dots, \tilde{U}_n) \end{aligned} \quad (3.33)$$

Using the similar procedure as that for  $\tilde{V} = f(\tilde{U})$  given in Subsection 3.1, the following theorem is obtained:

**Theorem 3.2** *Let  $\{\tilde{U}_i = (X_i, \{\tilde{U}_{i,\alpha} \mid \alpha \in (0, 1)\}, s_{\tilde{U}_i}) ; i = 1, \dots, n\}$  be a collection of fuzzy sets in  $\mathcal{G}(X_i)$ , and  $f: X_1 \times \dots \times X_n \rightarrow Y$  be a mapping from the product one  $X_1 \times \dots \times X_n$  of the metric space  $X_i$  to the metric space  $Y$ . Then,*

$$f(\tilde{U}_1, \tilde{U}_2, \dots, \tilde{U}_n) = (Y, \{f(\tilde{U}_{1,\alpha}, \dots, \tilde{U}_{n,\alpha}) \mid \alpha \in (0, 1)\}, s_{f(\tilde{v}_1, \dots, \tilde{v}_n)}) \quad (3.34)$$

*is the fuzzy set satisfying*

$$L_\alpha f(\tilde{U}_1, \dots, \tilde{U}_n) = f(L_\alpha \tilde{U}_1, \dots, L_\alpha \tilde{U}_n) \quad \text{for } \forall \alpha \in [0, 1] \quad (3.35a)$$

*and*

$$f(L_{\bar{\alpha}} \tilde{U}_1, \dots, L_{\bar{\alpha}} \tilde{U}_n) \subseteq L_{\bar{\alpha}} f(\tilde{U}_1, \dots, \tilde{U}_n) \quad \text{for } \forall \alpha \in (0, 1]. \quad (3.35b)$$

*Furthermore, if  $f$  is a continuous mapping,  $f(\tilde{U}_1, \tilde{U}_2, \dots, \tilde{U}_n)$  is the*

element of  $\mathcal{G}(Y)$ , and the equality in (3. 35b) holds, i. e.,

$$L_{\bar{\alpha}}f(\tilde{U}_1, \dots, \tilde{U}_n) = f(L_{\bar{\alpha}}\tilde{U}_1, \dots, L_{\bar{\alpha}}\tilde{U}_n) \quad \text{for } \forall \alpha \in (0, 1]. \quad (3. 36)$$

The proofs of Theorems 3. 1 and 3. 2 are proved in Appendix B.

#### 4 Arithmetic Means and Variances

Let  $\tilde{U}_i = (X, \{\tilde{U}_{i,\alpha} \mid \alpha \in (0, 1)\}, s_{\tilde{U}_i})$  ( $i=1, \dots, N$ ) be a sequence of fuzzy sets. Then, the arithmetic mean of  $\{\tilde{U}_i; i=1, \dots, N\}$  is heuristically given by

$$\begin{aligned} m(\tilde{U}_1, \dots, \tilde{U}_N) &= (X, \{(m(\tilde{U}_1, \dots, \tilde{U}_N))_{\alpha} \mid \alpha \in (0, 1)\}, s_{m(\tilde{U}_1, \dots, \tilde{U}_N)}) \\ &= \frac{1}{N} \sum_{i=1}^N \tilde{U}_i, \end{aligned} \quad (4. 1)$$

where  $s_{m(\tilde{U}_1, \dots, \tilde{U}_N)}$  is the predicate given through

$$s_{m(\tilde{U}_1, \dots, \tilde{U}_N)}(v) = \left\{ v \text{ coincides with } \frac{1}{N} \sum_{i=1}^N u_{i,0} \right\}. \quad (4. 2)$$

For the arithmetic mean  $m(\tilde{U}_1, \dots, \tilde{U}_N)$ , we have the following proposition :

**Proposition 4. 1** *The arithmetic mean  $m(\tilde{U}_1, \dots, \tilde{U}_N)$  given by (4. 1) is the element of  $\mathcal{G}(X)$  when each member  $\tilde{U}_i$  is the element of  $\mathcal{G}(X)$ , and its set representation is given through*

$$(m(\tilde{U}_1, \dots, \tilde{U}_N))_{\alpha} = \frac{1}{N} \sum_{i=1}^N \tilde{U}_{i,\alpha} \quad \text{for each } \alpha \in (0, 1). \quad (4. 3)$$

*The strong cut and level sets are given respectively by*

$$L_\alpha m(\tilde{U}_1, \dots, \tilde{U}_N) = \frac{1}{N} \sum_{i=1}^N L_\alpha \tilde{U}_i \quad \text{for each } \alpha \in [0, 1) \quad (4.4a)$$

and

$$L_{\bar{\alpha}} m(\tilde{U}_1, \dots, \tilde{U}_N) = \frac{1}{N} \sum_{i=1}^N L_{\bar{\alpha}} \tilde{U}_i \quad \text{for each } \alpha \in (0, 1]. \quad (4.4b)$$

Furthermore, if  $\tilde{U}_i \in \mathcal{G}_c(X)$  ( $i=1, \dots, N$ ),  $m(\tilde{U}_1, \dots, \tilde{U}_N)$  is an element of  $\mathcal{G}_c(X)$ .

The proof of Proposition 4.1 is given in Appendix C

In the following of this section, we restrict the basic space  $X$  to the ( $n$ -dimensional) Euclidean space  $R^n$ . Furthermore, the fuzzy sets in this section will be also restricted to those in  $\mathcal{G}_c(R^n)$ . Fuzzy sets in  $\mathcal{G}_c(R^n)$  are called "fuzzy vectors" in this paper. Hereafter, fuzzy vectors are represented by the lowercase letters with tilde like  $\tilde{u}$ ,  $\tilde{v}$ , etc.

Let  $\{\tilde{x}_i; 1, 2, \dots, N\}$  be a series of fuzzy vectors in  $\mathcal{G}_c(R^n)$ . Then, using Proposition 4.1, it is clear that the arithmetic mean of  $\{\tilde{x}_i\}$  given by

$$m(\tilde{x}_1, \dots, \tilde{x}_N) = \frac{1}{N} \sum_{i=1}^N \tilde{x}_i \quad (4.5)$$

is a fuzzy vector in  $\mathcal{G}_c(R^n)$  with

$$(m(\tilde{x}_1, \dots, \tilde{x}_N))_\alpha = \frac{1}{N} \sum_{i=1}^N \tilde{x}_{i,\alpha} \quad \text{for each } \alpha \in (0, 1), \quad (4.6)$$

$$L_\alpha m(\tilde{x}_1, \dots, \tilde{x}_N) = \frac{1}{N} \sum_{i=1}^N L_\alpha \tilde{x}_i \quad \text{for each } \alpha \in [0, 1) \quad (4.7a)$$

and

$$L_{\bar{\alpha}} m(\tilde{x}_1, \dots, \tilde{x}_N) = \frac{1}{N} \sum_{i=1}^N L_{\bar{\alpha}} \tilde{x}_i \quad \text{for each } \alpha \in (0, 1]. \quad (4.7b)$$

Next, Consider the statement  $s_{\text{var.}(\tilde{x}_1, \dots, \tilde{x}_N)}(M)$  such that

$$s_{\text{var.}(\tilde{x}_1, \dots, \tilde{x}_N)}(M) = \{M \text{ coincides with } \text{var.}(x_{1,o}, \dots, x_{N,o})\}, \quad (4.8)$$

where  $\text{var.}(x_{1,o}, \dots, x_{N,o})$  is the variance of  $\{x_{i,o}; i=1, \dots, N\}$ , i. e.,

$$\begin{aligned} \text{var.}(x_{1,o}, \dots, x_{N,o}) &= \frac{1}{N} \sum_{i=1}^N (x_{i,o} - m(x_{1,o}, \dots, x_{N,o})) \\ &\quad \times (x_{i,o} - m(x_{1,o}, \dots, x_{N,o}))' \\ &= \frac{1}{N} \sum_{i=1}^N x_{i,o} x_{i,o}' - \left( \frac{1}{N} \sum_{i=1}^N x_{i,o} \right) \left( \frac{1}{N} \sum_{i=1}^N x_{i,o} \right)'; \end{aligned} \quad (4.9)$$

and  $x_{i,o}$  is the original vector<sup>†</sup> of the fuzzy vector  $\tilde{x}_i = (R^n, \{\tilde{x}_{i,\alpha} \mid \alpha \in (0, 1)\}, s_{x_i})$ . Then, as a kind of the fuzzy variance of  $\{\tilde{x}_i; i=1, \dots, N\}$ , we have

$$\begin{aligned} &\text{var.}(\tilde{x}_1, \dots, \tilde{x}_N) \\ &= (R^{n \times n}, \{(\text{var.}(\tilde{x}_1, \dots, \tilde{x}_N))_{\alpha} \mid \alpha \in (0, 1)\}, s_{\text{var.}(\tilde{x}_1, \dots, \tilde{x}_N)}), \end{aligned} \quad (4.10)$$

where

$$(\text{var.}(\tilde{x}_1, \dots, \tilde{x}_N))_{\alpha} = \text{var.}(\tilde{x}_{1,\alpha}, \dots, \tilde{x}_{N,\alpha}) \quad (4.11)$$

and

$$\begin{aligned} \text{var.}(\tilde{x}_{1,\alpha}, \dots, \tilde{x}_{N,\alpha}) &= \{z \mid z = \text{var.}(x_1, \dots, x_N) \\ &\quad \text{with } x_i \in \tilde{x}_{i,\alpha} \text{ for each } i=1, \dots, N\}, \end{aligned} \quad (4.12)$$

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†) We call the original point of the fuzzy vector the original vector.

where  $\text{var.}(x_1, \dots, x_N)$  in (4.12) is given by (4.9) replacing  $x_{i_0}$  by  $x_i$ . Furthermore, applying Theorem 3.2 to (4.10), we have the following proposition:

**Proposition 4.2** *Let  $\{\tilde{x}_i; i=1, \dots, N\}$  be a sequence of fuzzy vectors. Then, the fuzzy variance given by (4.10) is a fuzzy set in  $\mathcal{G}(\mathbb{R}^{n \times n})$ , and satisfies*

$$L_\alpha(\text{var.}(\tilde{x}_1, \dots, \tilde{x}_N)) = \text{var.}(L_\alpha \tilde{x}_1, \dots, L_\alpha \tilde{x}_N) \quad \text{for } \forall \alpha \in [0, 1] \quad (4.13a)$$

and

$$L_{\bar{\alpha}}(\text{var.}(\tilde{x}_1, \dots, \tilde{x}_N)) = \text{var.}(L_{\bar{\alpha}} \tilde{x}_1, \dots, L_{\bar{\alpha}} \tilde{x}_N) \quad \text{for } \forall \alpha \in (0, 1]. \quad (4.13b)$$

## 5 Conclusions

In this paper, the some of basic properties of fuzzy sets obtained as vague perceptions of crisp phenomena have been investigated theoretically.

Adopting throughly the set representation approach for fuzzy sets, the basic properties of fuzzy sets and their functions have been studied with the help of the extension principle concept. Finally, some of basic properties of arithmetic means and variances for fuzzy vectors have been also investigated, which will be utilized for investigate statistical properties of fuzzy random vectors proposed by the author.

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## A Proofs of Theorems in Section 2

### Proof of Theorem 2. 2

**Proof of (2. 14a):** Using Theorem 2. 1, we have

$$L_\alpha \tilde{U} = \sup\{\beta \mid (\beta \in (0, 1)) \wedge (x \in \tilde{U}_\beta)\} > \alpha \quad (\text{A. 1})$$

when  $x \in L_\alpha \tilde{U}$ . Therefore, there exists  $\beta \in (\alpha, 1)$  such that  $x \in \tilde{U}_\beta$  and this yields

$$x \in \bigcup_{\beta \in (\alpha, 1)} \tilde{U}_\beta \quad (\text{A. 2})$$

or equivalently

$$L_\alpha \tilde{U} \subset \bigcup_{\beta \in (\alpha, 1)} \tilde{U}_\beta. \quad (\text{A. 3})$$

To show the inclusion in the opposite direction, we choose an  $x$  such that

$$x \in \bigcup_{\beta \in (\alpha, 1)} \tilde{U}_\beta. \quad (\text{A. 4})$$

Then, there is an  $\beta \in (\alpha, 1)$  such that  $x \in \tilde{U}_\beta$ , and it follows

$$x \in \tilde{U}_\beta \subset L_\beta \tilde{U} \subset L_\alpha \tilde{U}, \quad (\text{A. 5})$$

or equivalently

$$\bigcup_{\beta \in (\alpha, 1)} \tilde{U}_\beta \subseteq L_\alpha \tilde{U}. \quad (\text{A. 6})$$

**Proof of (2. 14b):** Let  $x \in L_\alpha \tilde{U}$ . Then, it follows

$$L_\alpha \tilde{U} = \sup\{\beta \mid (\beta \in (0, 1)) \wedge (x \in \tilde{U}_\beta)\} \geq \alpha \quad (\text{A. 7})$$

and hence for any  $\beta$  such that  $\beta \in (0, \alpha)$ , we have  $x \in \tilde{U}_\beta$  which means

$$x \in \bigcap_{\beta \in (0, \alpha)} \tilde{U}_\beta \quad (\text{A. 8})$$

or equivalently

$$L_{\bar{\alpha}} \tilde{U} \subset \bigcap_{\beta \in (0, \alpha)} \tilde{U}_\beta. \quad (\text{A. 9})$$

Conversely, assume that

$$x \in \bigcap_{\beta \in (0, \alpha)} \tilde{U}_\beta. \quad (\text{A. 10})$$

Then, it follows  $x \in \tilde{U}_\beta$  for any  $\beta$  such that  $\beta \in (0, \alpha)$  and therefore

$$(\tilde{U})(x) = \sup\{\beta \mid (\beta \in (0, 1)) \wedge (x \in \tilde{U}_\beta)\} \geq \alpha \quad (\text{A. 11})$$

which means  $x \in L_{\bar{\alpha}} \tilde{U}$  or equivalently

$$\bigcap_{\beta \in (0, \alpha)} \tilde{U}_\beta \subseteq L_{\bar{\alpha}} \tilde{U}. \quad (\text{A. 12})$$

### Proof of Theorem 2. 3

(i) Let  $\tilde{U}_\alpha$  be an arbitrary element of the set representation of  $\tilde{U}$ . Then,  $\tilde{U}_\alpha$  is a compact subset of the metric Space  $X$ , which means  $\tilde{U}_\alpha$  is the closed subset of  $X$ , and hence

$$L_{\bar{\alpha}} \tilde{U} = \bigcap_{\beta \in (0, \alpha)} \tilde{U}_\beta \quad (\text{A. 13})$$

is also closed. On the other hand, it can be verified from (2. 14a) that

$$\text{supp. } \tilde{U} = \text{cl. } \bigcup_{\alpha \in (0, 1)} \tilde{U}_\alpha = \text{cl. } L_0 \tilde{U}. \quad (\text{A. 14})$$

It is obvious that

$$\begin{aligned}
L_{\bar{\alpha}}\tilde{U} &= \{x \mid (\tilde{U})(x) \geq \alpha\} \subseteq \{x \mid (\tilde{U})(x) > 0\} \\
&= L_0\tilde{U} \subseteq \text{cl. } L_0\tilde{U} = \text{supp. } \tilde{U} \quad \text{for } \alpha \in (0, 1),
\end{aligned} \tag{A.15}$$

which means

$$L_{\bar{\alpha}}\tilde{U} \subseteq \text{supp. } \tilde{U}. \tag{A.16}$$

Therefore, we can conclude  $L_{\bar{\alpha}}\tilde{U}$  is compact because of its closeness and the compactness of  $\text{supp. } \tilde{U}$ .

(ii) Assume that  $\tilde{U} \in \mathcal{G}_c(X)$ . Then, using (2.14b), the  $\alpha$ -level set of  $\tilde{U}$  is given by

$$L_{\bar{\alpha}}\tilde{U} = \bigcap_{\beta \in (0, \alpha)} \tilde{U}_{\beta}, \tag{A.17}$$

and hence  $L_{\bar{\alpha}}\tilde{U}$  is convex for  $\forall \alpha \in (0, 1]$  because of those properties of  $\tilde{U}_{\beta}$  for  $\forall \beta \in (0, 1)$ . Furthermore, from (2.3), (2.4a) and (2.4b), it is easily verified that

$$L_{\beta_1}\tilde{U} \subseteq \tilde{U}_{\beta_1} \subseteq L_{\bar{\beta}_1}\tilde{U} \subseteq L_{\beta_2}\tilde{U} \subseteq \tilde{U}_{\beta_2} \subseteq L_{\bar{\beta}_2}\tilde{U} \quad \text{for } \beta_1 \geq \beta_2 > 0, \tag{A.18}$$

from which we have

$$\tilde{U}_{\beta_1} \subseteq \tilde{U}_{\beta_2} \quad \text{for } \beta_1 \geq \beta_2 > 0. \tag{A.19}$$

Let  $x_1$  and  $x_2$  be the elements of  $L_{\alpha}\tilde{U}$ . Then, there are  $\tilde{U}_{\beta_1}$  and  $\tilde{U}_{\beta_2}$  such that

$$x_1 \in \tilde{U}_{\beta_1} \quad \text{and} \quad x_2 \in \tilde{U}_{\beta_2} \tag{A.20}$$

because of

$$L_{\alpha}\tilde{U} = \bigcup_{\beta \in (\alpha, 1)} \tilde{U}_{\beta}, \tag{A.21}$$

which is given by (2. 14a). Hence, setting  $\beta_{\min} = \min(\beta_1, \beta_2)$  and  $\beta_{\max} = \max(\beta_1, \beta_2)$ , it follows from (A. 19) that

$$\tilde{U}_{\beta_{\max}} \subseteq \tilde{U}_{\beta_{\min}}. \quad (\text{A. 22})$$

Therefore, it is clear that

$$x = tx_1 + (1-t)x_2 \in \tilde{U}_{\beta_{\min}} \subseteq \bigcup_{\beta \in (\alpha, 1)} \tilde{U}_{\beta} = L_{\alpha} \tilde{U} \quad \text{for } \forall t \in [0, 1], \quad (\text{A. 23})$$

which means  $L_{\alpha} \tilde{U}$  is convex for  $\forall \alpha \in [0, 1)$ .

(iii) Assume that  $\tilde{U} \in \mathcal{G}_c(X)$ . Then, from the result of (ii), we have

$$L_0 \tilde{U} = \bigcup_{\alpha \in (0, 1)} \tilde{U}_{\alpha} \quad (\text{A. 24})$$

is convex. Then, we know that  $\text{cl. } L_0 \tilde{U}$  is also convex, which means so is  $\text{supp. } \tilde{U}$ .

## B Proofs of Theorems in Section 3

### Proof of Theorem 3. 1

(i) Proof of (3. 20): As given by (3. 14) and from its derivation process, it is obvious that  $f(L_{\bar{\alpha}} \tilde{U}) \subseteq L_{\bar{\alpha}} f(\tilde{U})$  for  $\alpha \in (0, 1]$ . Hence, in order to show (3. 20), it is sufficient to prove

$$f(L_{\bar{\alpha}} \tilde{U}) \supseteq L_{\bar{\alpha}} f(\tilde{U}) \quad \text{for } \alpha \in (0, 1]. \quad (\text{B. 1})$$

Let define  $M_{f; X}(y)$  by

$$M_{f; X}(y) = \{(\tilde{U})(x) \mid (y = f(x)) \wedge (x \in X)\}. \quad (\text{B. 2})$$

Then,  $L_{\bar{\alpha}} f(\tilde{U})$  may be rewritten by

$$L_{\bar{\alpha}}f(\tilde{U}) = \{\eta \in Y \mid \sup M_{f, X}(\eta) \geq \alpha\}. \quad (\text{B.3})$$

If  $y \in L_{\bar{\alpha}}f(\tilde{U})$  holds, then, we obtain

$$\sup M_{f, X}(y) \geq \alpha. \quad (\text{B.4})$$

Assume here that there is no  $x$  such that  $y = f(x)$  with  $x \in L_{\bar{\alpha}}\tilde{U}$ . Then, it should be that

$$\beta < \alpha \quad \text{for } \forall \beta \in M_{f, X}(y). \quad (\text{B.5})$$

Therefore,  $\alpha$  is an accumulation point of  $M_{f, X}(y)$ , i.e., there is a strictly increasing sequence  $\{\alpha_k \mid k=1, 2, \dots\}$  of elements  $\alpha_k \in M_{f, X}(y)$  with

$$\alpha_k \uparrow \alpha \quad \text{as } k \rightarrow \infty. \quad (\text{B.6})$$

The property  $\alpha_k \in M_{f, X}(y)$  is connected with the existence of  $x_k \in X$ , which satisfies the condition  $y = f(x_k)$  and  $(\tilde{U})(x_k) = \alpha_k$ , and hence

$$x_k \in L_{\bar{\alpha}_k}\tilde{U}. \quad (\text{B.7})$$

Since we have assumed  $\tilde{U} \in \mathcal{G}(X)$ ,  $L_{\bar{\alpha}_k}\tilde{U}$  is a compact subset of  $X$  for each  $k=1, 2, \dots$  as shown in Theorem 2.3. Therefore, we can conclude that the sequence  $\{x_k \mid k=1, 2, \dots\}$  has an accumulation point  $x$  and especially that it contains a partial sequence  $\{x_{k_j} \mid j=1, 2, \dots\}$  convergent to  $x$ . Furthermore, identifying  $L_{\bar{\alpha}_{k_j}}\tilde{U}$  as  $\tilde{U}_{\alpha_{k_j}}$ , it follows from (2.14b) that

$$x \in \bigcap_{j=1}^{\infty} L_{\bar{\alpha}_{k_j}}\tilde{U} = \bigcap_{\beta \in (0, \alpha)} L_{\bar{\beta}}\tilde{U} = L_{\bar{\alpha}}\tilde{U}. \quad (\text{B.8})$$

Since  $f$  is the continuous mapping, we obtain

$$\lim_{j \rightarrow \infty} x_{k_j} = x \Rightarrow \lim_{j \rightarrow \infty} f(x_{k_j}) = f(x) = y. \quad (\text{B. 9})$$

Hence,  $f(x) = y$  with  $x \in L_{\tilde{\alpha}} \tilde{U}$  as shown in (B. 8), which means

$$x \in \{w \in X \mid (y = f(w)) \wedge (w \in L_{\tilde{\alpha}} \tilde{U})\} \neq \emptyset, \quad (\text{B. 10})$$

which is a contradiction. Therefore, there exists an element  $x$  such that  $y = f(x)$  with  $x \in L_{\tilde{\alpha}} \tilde{U}$ , and this implies (B. 1) holds.

(ii) Proof of  $f(\tilde{U}) \in \mathcal{G}(Y)$ : It is obvious that  $f(\tilde{U}_{\alpha})$  is compact because of the continuity of  $f$ . Furthermore, using the equalities given by (3. 18), and (3. 20) proved in (i), it follows that

$$\bigcap_{\alpha \in (0, 1)} f(\tilde{U}_{\alpha}) = L_{\top} f(\tilde{U}) = f(L_{\top} \tilde{U}) = f\left(\bigcap_{\alpha \in (0, 1)} \tilde{U}_{\alpha}\right) \neq \emptyset \quad (\text{B. 11a})$$

and

$$\begin{aligned} \text{supp. } f(\tilde{U}) &= \text{cl. } \bigcup_{\alpha \in (0, 1)} (f(\tilde{U}))_{\alpha} = \text{cl. } L_0 f(\tilde{U}) = \text{cl. } f(L_0 \tilde{U}) \\ &\subseteq \text{cl. } f(\text{cl. } L_0 \tilde{U}) = f(\text{cl. } L_0 \tilde{U}) = f(\text{supp. } \tilde{U}). \end{aligned} \quad (\text{B. 11b})$$

Since  $f(\text{supp. } \tilde{U})$  is compact because of that of  $\text{supp. } \tilde{U}$  and the continuity of  $f$ , and consequently  $\text{supp. } f(\tilde{U})$  is also compact because of its closeness and the compactness of  $f(\text{supp. } \tilde{U})$ . Therefore, we can conclude  $f(\tilde{U}) \in \mathcal{G}(X)$ .

### Proof of Theorem 3. 2

(i) Proof of (3. 35a): The inclusion of  $L_{\alpha} f(\tilde{U}_1, \dots, \tilde{U}_n)$  to  $f(L_{\alpha} \tilde{U}_1, \dots, L_{\alpha} \tilde{U}_n)$  for  $\alpha \in (0, 1)$  has already been shown in (3. 33), and that for  $\alpha = 0$  is also obvious from the derivation process of (3. 33).

Therefore, in order to show (3. 35a), it is sufficient to show

$$f(L_\alpha \tilde{U}_1, \dots, L_\alpha \tilde{U}_n) \subseteq L_\alpha f(\tilde{U}_1, \dots, \tilde{U}_n) \quad \text{for } \alpha \in [0, 1). \quad (\text{B. 12})$$

Assume that

$$\begin{aligned} \eta &\in f(L_\alpha \tilde{U}_1, \dots, L_\alpha \tilde{U}_n) \\ &= \left\{ y \mid (y = f(x_1, \dots, x_n)) \wedge \left( \bigwedge_{i=1}^n (x_i \in L_\alpha \tilde{U}_i) \right) \right\} \\ &= \left\{ y \mid (y = f(x_1, \dots, x_n)) \wedge \left( \bigwedge_{i=1}^n ((\tilde{U}_i)(x_i) > \alpha) \right) \right\} \end{aligned} \quad (\text{B. 13})$$

Then, we can find that there is at least one collection  $(\zeta_1, \dots, \zeta_n)$  such that

$$(\zeta_1, \dots, \zeta_n) \in (L_\alpha \tilde{U}_1) \times \dots \times (L_\alpha \tilde{U}_n)$$

satisfying  $\eta = f(\zeta_1, \dots, \zeta_n)$ , which yields

$$\begin{aligned} \eta = f(\zeta_1, \dots, \zeta_n) &\in \left\{ y \mid (y = f(x_1, \dots, x_n)) \wedge \left( \bigwedge_{i=1}^n ((\tilde{U}_i)(x_i) > \alpha) \right) \right\} \\ &\subseteq \left\{ y \mid \sup \left\{ \min_{i=1, \dots, n} (\tilde{U}_i)(x) \mid (y = f(x_1, \dots, x_n)) \wedge \left( \bigwedge_{i=1}^n (x_i \in X_i) \right) \right\} > \alpha \right\} \\ &= \left\{ y \mid t(s_{f(\tilde{U}_1, \dots, \tilde{U}_n)}(y)) > \alpha \right\} = L_\alpha f(\tilde{U}_1, \dots, \tilde{U}_n), \end{aligned} \quad (\text{B. 14})$$

which is equivalent to (B. 12).

(ii) Proof of (3. 35b): As given by (3. 33) and from its derivation process, it is obvious that

$$f(L_{\bar{\alpha}} \tilde{U}_1, \dots, L_{\bar{\alpha}} \tilde{U}_n) \subseteq L_{\bar{\alpha}} f(\tilde{U}_1, \dots, \tilde{U}_n) \quad (\text{B. 15})$$

for  $\alpha \in (0, 1]$ . Tence, in order to show (3. 35b), it is sufficient to prove

$$f(L_{\bar{\alpha}}\tilde{U}_1, \cdots, L_{\bar{\alpha}}\tilde{U}_n) \supseteq L_{\bar{\alpha}}f(\tilde{U}_1, \cdots, \tilde{U}_n) \quad \text{for } \alpha \in (0, 1]. \quad (\text{B. 16})$$

Let define  $M_{f; x_1, \cdots, x_n}(y)$  by

$$M_{f; x_1, \cdots, x_n}(y) = \left\{ \min_{i=1, \cdots, n} (\tilde{U}_i)(x_i) \mid (y=f(x_1, \cdots, x_n)) \right. \\ \left. \wedge \left( \bigwedge_{i=1}^n (x_i \in X_i) \right) \right\}. \quad (\text{B. 17})$$

Then,  $L_{\bar{\alpha}}f(\tilde{U}_1, \cdots, \tilde{U}_n)$  may be rewritten by

$$L_{\bar{\alpha}}f(\tilde{U}_1, \cdots, \tilde{U}_n) = \{ \eta \in Y \mid \sup M_{f; x_1, \cdots, x_n}(\eta) \geq \alpha \}. \quad (\text{B. 18})$$

If  $y \in L_{\bar{\alpha}}f(\tilde{U}_1, \cdots, \tilde{U}_n)$  holds, then, we obtain

$$\sup M_{f; x_1, \cdots, x_n}(y) \geq \alpha. \quad (\text{B. 19})$$

Assume here that there is no  $(x_1, \cdots, x_n)$  such that  $y=f(x_1, \cdots, x_n)$  with  $x_i \in L_{\bar{\alpha}}\tilde{U}_i$  for each  $i=1, \cdots, n$ . Then, it should be that

$$\beta < \alpha \quad \text{for all } \beta \in M_{f; x_1, \cdots, x_n}(y). \quad (\text{B. 20})$$

Therefore,  $\alpha$  is an accumulation point of  $M_{f; x_1, \cdots, x_n}(y)$ , i. e., there is a strictly increasing sequence  $\{\alpha_k \mid k=1, 2, \cdots\}$  of elements  $\alpha_k \in M_{f; x_1, \cdots, x_n}(y)$  with

$$\alpha_k \uparrow \alpha \quad \text{as } k \rightarrow \infty. \quad (\text{B. 21})$$

The property  $\alpha_k \in M_{f; x_1, \cdots, x_n}(y)$  is connected with the existence of  $x_{i, k} \in X_i$  for each  $i=1, \cdots, n$  which satisfies the condition  $y=f(x_{1, k}, \cdots, x_{n, k})$  and  $(\tilde{U}_i)(x_{i, k})=\alpha_k$  for each  $i=1, \cdots, n$ , and hence

$$x_{i,k} \in L_{\bar{a}_k} \tilde{U}_i \quad \text{for each } i=1, \dots, n. \quad (\text{B. 22})$$

Since we have assumed  $\tilde{U}_i \in \mathcal{G}(X_i)$ ,  $L_{\bar{a}_k} \tilde{U}_i$  is a compact subset of  $X_i$  as shown in Theorem 2.3. Therefore, using Tychonoff's theorem, it can be shown that

$$(L_{\bar{a}_k} \tilde{U}_1) \times \dots \times (L_{\bar{a}_k} \tilde{U}_n) \quad (\text{B. 23})$$

is also a compact subset of  $X_1 \times \dots \times X_n$  for each  $k=1, 2, \dots$ . Therefore, we can conclude that the sequence  $\{x_k = (x_{1,k}, \dots, x_{n,k}) \mid k=1, 2, \dots\}$  has an accumulation point  $x = (x_1, \dots, x_n)$  and especially that it contains a partial sequence  $\{x_{k_j} \mid j=1, 2, \dots\}$  convergent to  $x$ . Furthermore, identifying  $L_{\bar{a}_k} \tilde{U}_i$  as  $\tilde{U}_{i, \bar{a}_k}$ , it follows from (2. 14b) that

$$\begin{aligned} x &\in \left( \bigcap_{j=1}^{\infty} L_{\bar{a}_{k_j}} \tilde{U}_1 \right) \times \dots \times \left( \bigcap_{j=1}^{\infty} L_{\bar{a}_{k_j}} \tilde{U}_n \right) \\ &= \left( \bigcap_{\beta \in (0, \alpha)} L_{\bar{\beta}} \tilde{U}_1 \right) \times \dots \times \left( \bigcap_{\beta \in (0, \alpha)} L_{\bar{\beta}} \tilde{U}_n \right) \\ &= (L_{\bar{\alpha}} \tilde{U}_1) \times \dots \times (L_{\bar{\alpha}} \tilde{U}_n). \end{aligned} \quad (\text{B. 24})$$

Since  $f$  is the continuous mapping, we obtain

$$\lim_{j \rightarrow \infty} x_{k_j} = x \Rightarrow \lim_{j \rightarrow \infty} f(x_{1,k_j}, \dots, x_{n,k_j}) = f(x_1, \dots, x_n) = y. \quad (\text{B. 25})$$

Hence,  $f(x_1, \dots, x_n) = y$  with

$$x = (x_1, \dots, x_n) \in (L_{\bar{\alpha}} \tilde{U}_1) \times \dots \times (L_{\bar{\alpha}} \tilde{U}_n) \quad (\text{B. 26})$$

as shown in (B. 24), and (B. 26) shows

$$(x_1, \dots, x_n) \in \left\{ (w_1, \dots, w_n) \in X_1 \times \dots \times X_n \mid (y = f(w_1, \dots, w_n)) \right. \\ \left. \wedge \left( \bigcap_{i=1}^n (w_i \in L_{\tilde{a}} \tilde{U}_i) \right) \right\} \neq \emptyset, \quad (\text{B. 27})$$

which is a contradiction. Therefore, there exists an element  $(x_1, \dots, x_n)$  such that  $y = f(x_1, \dots, x_n)$  with  $x_i \in L_{\tilde{a}} \tilde{U}_i$  for each  $i = 1, \dots, n$ , which means (B. 15) holds.

(iii) Proof of  $f(\tilde{U}_1, \dots, \tilde{U}_n) \in \mathcal{G}(Y)$ : It is obvious that  $f(\tilde{U}_{1,\alpha}, \dots, \tilde{U}_{n,\alpha})$  is compact because of the continuity of  $f$ . Furthermore, using the equalities given by (3. 35a), and (3. 35b) proved in (i), it follows that

$$\bigcap_{\alpha \in (0, 1)} f(\tilde{U}_{1,\alpha}, \dots, \tilde{U}_{n,\alpha}) = L_{\top} f(\tilde{U}_1, \dots, \tilde{U}_n) = f(L_{\top} \tilde{U}_1, \dots, L_{\top} \tilde{U}_n) \\ = f\left(\bigcap_{\alpha \in (0, 1)} \tilde{U}_{1,\alpha}, \dots, \bigcap_{\alpha \in (0, 1)} \tilde{U}_{n,\alpha}\right) \neq \emptyset \quad (\text{B. 28a})$$

and

$$\text{supp. } f(\tilde{U}_1, \dots, \tilde{U}_n) = \text{cl. } \bigcup_{\alpha \in (0, 1)} (f(\tilde{U}_1, \dots, \tilde{U}_n))_{\alpha} \\ = \text{cl. } L_0 f(\tilde{U}_1, \dots, \tilde{U}_n) \\ = \text{cl. } f(L_0 \tilde{U}_1, \dots, L_0 \tilde{U}_n) \\ \subseteq \text{cl. } f(\text{cl. } L_0 \tilde{U}_1, \dots, \text{cl. } L_0 \tilde{U}_n) \\ = f(\text{cl. } L_0 \tilde{U}_1, \dots, \text{cl. } L_0 \tilde{U}_n) \\ = f(\text{supp. } \tilde{U}_1, \dots, \text{supp. } \tilde{U}_n). \quad (\text{B. 28b})$$

Since  $f(\text{supp. } \tilde{U}_1, \dots, \text{supp. } \tilde{U}_n)$  is compact because of that of  $(\text{supp. } \tilde{U}_1) \times \dots \times (\text{supp. } \tilde{U}_n)$  and the continuity of  $f$ . Hence,  $\text{supp. } f(\tilde{U}_1, \dots, \tilde{U}_n)$  is also compact because of its closeness and the compactness of  $f(\text{supp. } \tilde{U}_1, \dots, \text{supp. } \tilde{U}_n)$ . Therefore, we can conclude  $f(\tilde{U}_1, \dots, \tilde{U}_n) \in \mathcal{G}(Y)$ .

## C Proofs of Proposition in Section 4

### Proof of Proposition 4. 1

The property of  $m(\tilde{U}_1, \dots, \tilde{U}_N) \in \mathcal{G}(X)$  and those given by (4. 4 a) and (4. 4b) are verified as easy application of Theorem 3.2. Hence, only  $m(\tilde{U}_1, \dots, \tilde{U}_N) \in \mathcal{G}_c(X)$  will be shown when  $\tilde{U}_i \in \mathcal{G}_c(X)$  for each  $i=1, \dots, N$ .

Let  $\tilde{U}_i \in \mathcal{G}_c(X)$  and  $u_i^{(j)} \in \tilde{U}_{i, \alpha}$  for each  $i=1, \dots, N$  and  $j=1$  and  $2$ . Furthermore, let  $z, z^{(1)}$  and  $z^{(2)}$  be respectively given by

$$z = \lambda z^{(1)} + (1 - \lambda) z^{(2)} \quad \text{for } \forall \lambda \in [0, 1] \quad (\text{C. 1a})$$

$$z^{(j)} = \sum_{i=1}^N u_i^{(j)} \quad \text{for } j=1 \text{ and } 2. \quad (\text{C. 1b})$$

Then,  $z$  can be rewritten by

$$\begin{aligned} z &= \sum_{i=1}^N \lambda u_i^{(1)} + \sum_{i=1}^N (1 - \lambda) u_i^{(2)} \\ &= \sum_{i=1}^N (\lambda u_i^{(1)} + (1 - \lambda) u_i^{(2)}) \\ &= \sum_{i=1}^N u_i, \end{aligned} \quad (\text{C. 2})$$

where

$$u_i = \lambda u_i^{(1)} + (1 - \lambda) u_i^{(2)} \in \tilde{U}_{i, \alpha} \quad (\text{C. 3})$$

because of the convex property of  $\tilde{U}_{i, \alpha}$ , which implies  $z \in \sum_{i=1}^N \tilde{U}_{i, \alpha}$  and hence the convexity of  $\sum_{i=1}^N \tilde{U}_{i, \alpha}$  is proved. The convexity of  $(1/N) \sum_{i=1}^N \tilde{U}_{i, \alpha}$  is easily verified from that for  $\sum_{i=1}^N \tilde{U}_{i, \alpha}$ , and this means  $m(\tilde{U}_1, \dots, \tilde{U}_N) \in \mathcal{G}_c(X)$ .