Entropy conditions for solutions to a system of conservation laws

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§ 1.

Let us consider a 2×2 system of conservation laws

$$u_t + f(u)_x = 0, \quad (x, t) \in R \times (0, \infty) \tag{1.1}$$

where $u = {}^t(u_1, u_2)$, $f(u) = {}^t(f_1(u), f_2(u))$. We say that a pair of C^1-function $(\eta(u), q(u))$ is an entropy pair, if it satisfies

$$\text{grad } q(u) = \text{grad } \eta(u) f'(u). \tag{1.2}$$

If f is smooth and u is a Lipschitz solution to (1.1), u is also a solution to the associated quasi-linear system

$$u_t + f'(u) u_x = 0. \tag{1.3}$$

Hence u satisfies an additional conservation law

$$\eta(u)_t + q(u)_x = 0. \tag{1.4}$$

On the other hand, if $u \in L^1_{loc}(R \times [0, \infty))$ is a weak solution to (1.1), (1.3) does not always make sense; nor (1.4) is necessarily satisfied.

Definition 1.1. Let u be a weak solution to (1.1) and (η, q) be an entropy pair. Assume that η is a convex function of u. We say that u satisfies the entropy condition, if

$$\eta(u)_t + q(u)_x \leq 0 \tag{1.5}$$

holds in the weak sense for any such (η, q).

To proceed further, we require that the system (1.3) is hyperbolic. That is: the matrix $f'(u)$ has real distinct eigenvalues, which are denoted by $\lambda_1(u)$, $\lambda_2(u)$ $(\lambda_1 < \lambda_2)$.

Let us consider the following discontinuous solution.

$$u(x, t) = \begin{cases} u_L & x/t < s \\ u_R & x/t > s. \end{cases} \tag{1.6}$$

We know that $u(x, t)$ given by (1.6) is a weak solution to (1.1), if and only if u_L, u_R satisfy

$$s(u_R - u_L) = f(u_R) - f(u_L) \qquad \text{(Rankine-Hugoniot condition).} \tag{1.7}$$

Definition 1.2. We say that $u(x, t)$ given by (1.6) is a k-shock, if u_L, u_R satisfy

$$\lambda_k(u_R) \leq s \leq \lambda_k(u_L) \quad (k = 1, 2). \tag{1.8}$$

When the system (1.1) is genuinly nonlinear, Lax constructed, for each $k = 1, 2$, a 1-parameter family $u_R = u_k(\varepsilon)$, $(\varepsilon \leq 0)$ which satisfies $u_k(0) = u_L$ and the strict shock condition

$$\lambda_k(u_R) < s < \lambda_k(u_L) \quad (k = 1, 2) \qquad \text{(Lax shock condition, see [3]).} \tag{1.9}$$

Moreover he also proved in [4] that the weak solution (1.6) with (1.7) satisfies the shock condition (1.9), if and only if u satisfies

$$\eta(u_L) - \eta(u_R) - q(u_L) + q(u_R) < 0 \tag{1.10}$$

(Theorem 2.1 in [4]). Hence it is clear that all solution of (1.1) constructed by Glimm's method satisfy the entropy condition (1.5) (Theorem 2.1 in [4]).

Let us denote by $R_k(u)$ an eigenvector corresponding to $\lambda_k(u)$ $(k = 1, 2)$. Also we denote by w a 2-Riemann invariant and z a 1-Riemann invariant. The aim of this note is to construct two families of shock wave solutions $u_k(\varepsilon)$ $(k = 1, 2)$ under the following condition.

Condition 1.3

$$(1.11)\quad R_1 \cdot \operatorname{grad} \lambda_1 = \begin{cases} > 0 & z > 0 \\ = 0 & z = 0 \\ < 0 & z < 0 \end{cases}, \qquad R_2 \cdot \operatorname{grad} \lambda_2 = \begin{cases} > 0 & w > 0 \\ = 0 & w = 0 \\ < 0 & w < 0 \end{cases}.$$

We note that 1-D nonlinear Maxwell's equations satisfy the condition under suitable conditions (see [1]). We can also prove, in the same manner as [4], that the shock wave solution satisfies (1.8), if and only if it satisfies

$$(1.12)\qquad \eta(u_L) - \eta(u_R) - q(u_L) + q(u_R) \leq 0.$$

Hence the solutions obtained in [1] satisfy the entropy condition (1.5).

§ 2

First we claim that there exists a 1-parameter family of 1-shocks $\{u(\varepsilon)\}$ satisfying $u(0) = u_L$.

Proposition 2.1. *Let the system* (1.1) *be hyperbolic in* U *and* $u_L \in U$. *Then there exist smooth* 1*-parameter families* $\{u(\varepsilon)\}$, $\{s(\varepsilon)\}$ *defined for* $|\varepsilon| < \varepsilon_0$ *which satisfy*

$$(2.1)\qquad \begin{aligned} &s(\varepsilon)(u(\varepsilon) - u_L) = f(u(\varepsilon)) - f(u_L) \\ &u(0) = u_L,\ s(0) = \lambda_1(u_L). \end{aligned}$$

This proposition is contained in Theorem 17. in the text book [5]. $u(\varepsilon)$ and $s(\varepsilon)$ are obtained through the bifurcation of solutions to (2.1) from $R(u_L)$ and $\lambda_1(u_L)$.

Differentiating (2.1) with respect to ε once and twice and setting $\varepsilon = 0$, we find that

$$(2.2)\quad \lambda_1(u_L)\frac{du}{d\varepsilon}(0) = A(u_L)\frac{du}{d\varepsilon}(0)$$

$$(2.3)\quad 2\frac{ds}{d\varepsilon}(0)R_1(u_L) + \lambda_1(u)\frac{d^2u}{d\varepsilon^2}(0) = \operatorname{Hess}(f)\left[\frac{du}{d\varepsilon}, \frac{du}{d\varepsilon}\right](u_L) + A(u_L)\frac{d^2u}{d\varepsilon^2}.$$

(2.2) says that $(du/d\varepsilon)(0)$ is propotional to $R_1(u_L)$. Then changing the parametrization, if necessary, we have

$$\frac{du}{d\varepsilon}(0) = R_1(u_L). \tag{2.4}$$

In a similar manner, differentiating the identity

$$A(u(\varepsilon))R_1(u(\varepsilon)) = \lambda_1(u(\varepsilon))R_1(u(\varepsilon))$$

with respect to ε and setting $\varepsilon = 0$, we have

$$\frac{d\lambda_1}{d\varepsilon}(0)R_1(u_L)+\lambda_L(u_L)\frac{dR_1}{d\varepsilon}(0) = \mathrm{Hess}(f)\left[R_L, \frac{du}{d\varepsilon}\right](u_L)+A(u_L)\frac{dR_1}{d\varepsilon}(0). \tag{2.5}$$

Now subtract (2.5) from (2.3). Then we obtain

$$\left(2\frac{ds}{d\varepsilon}(0) - \frac{d\lambda_1}{d\varepsilon}(0)\right)R_1 = (A(u_L) - \lambda_1(u_L))\left(\frac{d^2u}{d\varepsilon^2}(0) - \frac{dR_1}{d\varepsilon}(0)\right). \tag{2.6}$$

Hence by multiplying L_1 to (2.6), we find that

$$\frac{ds}{d\varepsilon}(0) = \frac{1}{2}\frac{d\lambda_1}{d\varepsilon}(0) = \tfrac{1}{2}(\mathrm{grad}\,\lambda_1 \cdot R_1)\ (u_L). \tag{2.7}$$

Also we have

$$\frac{d^2u}{d\varepsilon^2}(0) = \frac{dR_1}{d\varepsilon}(0) + aR_1(u_L)$$

By the change of parametrization $\varepsilon \to \varepsilon + \tfrac{1}{2}a\varepsilon^2$, we have

$$\frac{d^2u}{d\varepsilon^2}(0) = (R_1 \cdot \mathrm{grad})R_1(u_L)\,. \tag{2.8}$$

Thus we obtain the following expansion for $u(\varepsilon)$ and $s(\varepsilon)$.

$$u(\varepsilon) = u_L + \varepsilon R_1(u_L) + \tfrac{1}{2}\varepsilon^2(R_1 \cdot \operatorname{grad})R_1(u_L) + 0(\varepsilon^3) \tag{2.9}$$

$$s(\varepsilon) = \lambda_1(u_L) + \tfrac{1}{2}\varepsilon(R_1 \cdot \operatorname{grad}\lambda_1)(u_L) + 0(\varepsilon^2) .$$

By the above expansions, we obtain the following propositions.

Proposition 2.2 (S_1: 1-shock). *Let $u_L \in U$. Then there exist 1-parameter families $\{u(\varepsilon)\}$, $\{s(\varepsilon)\}$ such that*

$$u(x, t) = \begin{cases} u_L & x/t < s(\varepsilon) \\ u(\varepsilon) & x/t > s(\varepsilon) \end{cases}$$

is a 1-shock with $u_R = u(\varepsilon)$.

Proof. It is obvious that u satisfies the Rankine-Hugoniot condition (i.e. u is a weak sulution). Let us assume $z(u_L) > 0$. Note that

$$\lambda_1(u(\varepsilon)) = \lambda_1(u_L) + \varepsilon(R_1 \cdot \operatorname{grad}\lambda_1)(u_L) + 0(\varepsilon^2) .$$

Then if $\varepsilon R_1 \cdot \operatorname{grad}\lambda_1 < 0$, we can see that $\lambda_1(u(\varepsilon)) < s(\varepsilon) < \lambda_1(u_L)$ (the Lax shock condition) holds for such ε as is small in comparison with $|R_1 \cdot \operatorname{grad}\lambda_1|$. Now, suppose that $\lambda_1(u(\varepsilon)) < s(\varepsilon)$ for $\varepsilon_0 < \varepsilon < 0$ and $\lambda_1(u(\varepsilon_0)) = s(\varepsilon_0)$. Differentiating the Rankine-Hugoniot condition with respect to ε, we have

$$\frac{ds}{d\varepsilon}(\varepsilon)\,(u(\varepsilon) - u_L) + s(\varepsilon)\frac{du}{d\varepsilon}(\varepsilon) = A(u(\varepsilon))\frac{du}{d\varepsilon}(\varepsilon) . \tag{2.10}$$

Hence multiplying L_1 to (2.21), we obtain

$$\frac{ds}{d\varepsilon}L_1(u)(u - u_L) + (s - \lambda_1(u))\,L_1(u)\frac{du}{d\varepsilon} = 0 . \tag{2.11}$$

Since $L_1(u)(u - u_L) \neq 0$, $L_1(u)(du/d\varepsilon) \neq 0$ for small $\varepsilon \neq 0$, (2.11) says that $s(\varepsilon) = \lambda(u(\varepsilon))$ if and only if $(ds/d\varepsilon)(\varepsilon) = 0$. In such a case, we have by (2.10)

$$\lambda_1(u(\varepsilon_0))\frac{du}{d\varepsilon}(\varepsilon_0) = A(u(\varepsilon_0))\frac{du}{d\varepsilon}(\varepsilon_0).$$

Thus we find that

$$\frac{du}{d\varepsilon}(\varepsilon_0) = aR_1(u(\varepsilon_0)) \ .$$

Since $du/d\varepsilon = R_1(u(\varepsilon)) + 0(\varepsilon)$, we may assume $a > 0$. If $z(u(\varepsilon_0)) > 0$, we have

$$\frac{d\lambda_1}{d\varepsilon}(u(\varepsilon_0)) = \frac{du}{d\varepsilon}(\varepsilon_0) \cdot \text{grad } \lambda(u(\varepsilon_0)) = aR(u(\varepsilon_0)) \cdot \text{grad } \lambda(u(\varepsilon_0)) > 0 \ .$$

Hence we find that $\lambda_1(\varepsilon) = \lambda_1(u(\varepsilon))$ is strictly increasing at ε_c, which is impossible. If $z(u(\varepsilon_0)) = 0$, we have $\lambda_1(u(\varepsilon_0)) = s(\varepsilon_0) = \lambda^0$. Let v is a constant vector satisfying $z(v) = 0$, $v - u_L = \varepsilon R_2(u_L) + 0(\varepsilon^2)$. Then it follows that

$$\begin{aligned} \lambda^0(v - u_L) &= \lambda^0(v - u(\varepsilon_0)) + \lambda^0(u(\varepsilon_0) - u_L) \\ &= f(v) - f(u(\varepsilon_0)) + f(u(\varepsilon_0)) - f(u_L) \\ &= f(v) - f(u_L) = A(u_L)\ (v - u_L) + 0(\varepsilon^2), \end{aligned}$$

which is also impossible. Thus we have

$$\lambda_1(u(\varepsilon)) < s(\varepsilon) \quad \text{for } \varepsilon < 0,$$

as long as $u(\varepsilon)$ lies in a neighborhood of 0.

Next suppose that $s(\varepsilon_0) = \lambda_1(u_L)$ for some $\varepsilon_0 < 0$. Since $s(0) = \lambda_1(u_L)$, there exists ε_1 such that $(ds/d\varepsilon)(\varepsilon_1) = 0$. Then (2.22) says that $s(\varepsilon_1) = \lambda_1(u(\varepsilon_1))$, which is not the case either. Thus we have proved that u satisfies the Lax shock condition and hence the solution is a 1-shock.

Let us now assume that $z(u_L) = 0$. Then we have $z(u(\varepsilon)) = 0$ and $ds/d\varepsilon = 0$ for all ε. Hence we find that u satisfies

$$\lambda_1(u_R) = \lambda_1(u_L) = \lambda^0$$

Thus we have proved the proposition.

§ 3

We first guarantee the existence of an entropy pair (η, q) in which η is strictly convex. We can see that there exists an entropy flux q, if and only if

η satisfies

$$\text{rot}(\text{grad } \eta(u)f'(u)) = 0. \tag{3.1}$$

(3.1) is a second order hyperbolic equation for η, if (1.3) is hyperbolic. Hence there exists a strictly convex solution η to (3.1) (see [4]). In particular, for 1-D nonlinear Maxwell's equations, we find that

$$\eta = ED - L(E, B), \quad q = EH$$

constitute an entropy pair.

Theorem 3.1. *Let u be a k-shock. Then u satisfies*

$$s[\eta(u_L) - \eta(u_R)] - q(u_L) + q(u_R) \leq 0 \tag{3.2}$$

for any entropy pair (η, q) in which η is strictly convex.

Proof. Let $u_R = u(\varepsilon)$ and $s = s(\varepsilon)$. We set

$$E(\varepsilon) = s(\varepsilon)[\eta(u_L) - \eta(u(\varepsilon))] - q(u_L) + q(u(\varepsilon)).$$

Differenciating E with respect to ε, we have

$$\begin{aligned}\frac{dE}{d\varepsilon} &= \frac{ds}{d\varepsilon}[\eta(u_L) - \eta(u(\varepsilon))] - s(\varepsilon)\eta'(\varepsilon)\frac{du}{d\varepsilon} + q'(\varepsilon)\frac{du}{d\varepsilon} \\ &= \frac{ds}{d\varepsilon}[\eta(u_L) - \eta(u(\varepsilon))] - \eta'(\varepsilon)\left[s(\varepsilon)\frac{du}{d\varepsilon} - f'(u(\varepsilon))\frac{du}{d\varepsilon}\right].\end{aligned}$$

By (2.10), we obtain

$$\frac{dE}{d\varepsilon} = \frac{ds}{d\varepsilon}[\eta(u_L) - \eta(u(\varepsilon)) - \eta'(u(\varepsilon))(u_L - u(\varepsilon))]. \tag{3.3}$$

Since η is strictly convex, we find that

$$\eta(u_L) - \eta(u(\varepsilon)) - \eta'(u(\varepsilon))(u_L - u(\varepsilon)) > 0.$$

When $z(u_L) > 0$, we have shown in §2, that $(ds/d\varepsilon)(0) > 0$ and $ds/d\varepsilon \neq 0$.

Hence $ds/d\varepsilon > 0$ and then $dE/d\varepsilon > 0$. If $z(u_L) < 0$, we have similarly $dE/d\varepsilon < 0$. Thus we have $E(\varepsilon) < 0$ in both cases. If $z(u_L) = 0$, we have obviously $ds/d\varepsilon = 0$. Then we have $E = 0$. Thus we have proved the theorem.

Let $u = u_{\theta, h}$ be an approximate sulution obtained in [1] by Glimm's method ([2]). Theorem 3.1 says that $u_{\theta, h}$ satisfies

$$\eta(u_{\theta, h})_t + q(u_{\theta, h})_x \leq 0$$

in each strip $(n-1)k \leq t < nk$. Hence for $\phi > 0$ with compact support, we find that

$$\begin{aligned}
&-\int_0^\infty \int_R [\phi_t \eta(u_{\theta, h}) + \phi_x q(u_{\theta, h})]\, dxdt \\
&= \sum_{n \geq 0} \int_R \phi(x, nk)\ [\eta u_{\theta, h}(x, nk) - \eta(u_{\theta, h}(x, nk - 0))]\, dx \\
&\quad + \sum_{n \geq 0} \int_{nk}^{(n+1)k} \int_R \phi\ [\eta(u_{\theta, h})_t + q(u_{\theta, h})_x]\, dxdt \\
&\leq \sum_{n \geq 0} \int_R \phi(x, nk)\ [\eta(u_{\theta, h}(x, nk) - \eta(u_{\theta, h}(x, nk - 0))]\, dx\ .
\end{aligned}$$

Since it is shown in [1], [2] that the last sum tends to zero for a suitably selected subsequence h_j. Thus we have the following theorem.

Theorem 3.2. *All weak solutions constructed by Glimm's method satisfy the entropy inequality*

$$\eta(u)_t + q(u)_x \leq 0$$

for every entropy pair (η, q) in which η is strictly convex.

References

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