COLLECTIVE PREFERENCES UNDER UNCERTAINTY

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I. Introduction

In this paper, we shall consider a collective decision-making whose problem appears in the n-person game scheme. This problem has been studied by J. C. Harsanyi [2,3,4] who proved that there exists a social welfare function which was represented by the sum of each individual von Neumann-Morgenstern utility. Our object to study this problem is that we shall apply the multivariate von Neumann-Morgenstern utility theory [5, 6] and make it clear how the individual preferences are correlated with the social preferences. We have used the same definitions and notations in the papers [5, 6] .

II. Individual Preferences and Collective Preferences

We shall consider a game under uncertainty in which n persons participate. Suppose that a set of the game's sure consequences is denoted by a separable metric space Y and each player has a complete preordering $\overset{c}{\gtrsim}$ on Y, and a set of lottery tickets of the game is denoted by the metric space $M(Y)$ endowed with the topology of weak convergence and each player has a complete preordering $\overset{l}{\gtrsim}$ on a subset P of the metric space $M(Y)$.

Now each player' s *common rationality postulates* are given as follows;

Condition 1. *For any* $p^0 \in P$, *the sets* $\{p \in p \mid p \overset{l}{\gtrsim} p^0\}$ *and* $\{p \in p \mid p^0 \overset{l}{\gtrsim} p\}$ *are closed in P.*

Condition 2. *For any* $p^1, p^2, p^3 \in P$ *and any real number* $t \in [0, 1]$, $p^1 \overset{l}{\sim} p^2$ *implies* $tp^1 + (1-t)p^3 \overset{l}{\sim} tp^2 + (1-t)p^3$.

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Condition 3. *For any $y, y' \in Y$, $y \overset{c}{\gtrsim} y'$ is equivalent to $p_y \overset{l}{\gtrsim} p_{y'}$, where p_y and $p_{y'}$ belong to D.*

If each player satisfies the above three Conditions, we get the next Theorem.

Theorem 1. *Let Y be a separable metric space. Let $\overset{c}{\gtrsim}$ be a complete preordering defined on the metric space Y, and $\overset{l}{\gtrsim}$ a complete preordering defined on σ-convex subspace P of M(Y) such that $D \subset P$. Then, Conditions 1, 2 and 3 are necessary and sufficient for the existence of a real-valued continuous bounded function h defined on P by $h(p) = \int_Y u\, dp$ for any $p \in p$, where p is a probability measure on B(Y), is order-preserving with respect to $\overset{l}{\gtrsim}$ and σ-linear. Any h and u are unique up to a positive linear transformation. (For proof, see Grandmont, J. M., [1].)*

According to Theorem 1, each player should choose a lottery which maximizes his expected utility. Next, we shall consider how conditions are necessary and sufficient for a *collective utility* which may evaluate a n-tuple of lotteries that each player chooses. Then, we may apply the multivariate von Neumann-Morgenstern utility theory [5, 6] to this case.

A set of the n-tuple of sure consequences is denoted by a product separable metric space Y^n and a set of the n-tuple of each player' s lottery ticket, by a product metric space $M^n(Y)$ endowed with the topology of weak convergence. Then we may be faced with difficulties how we should define *collective preorderings* $\overset{c}{\gtrsim}$ and $\overset{l}{\gtrsim}$ on Y^n and a subset P of $M^n(Y)$, resectively. Thus, we shall consider *product preorderings* which are defined by $\overset{c}{\gtrsim} = \prod_{i=1}^{n} \overset{c}{\gtrsim}_i$ and $\overset{l}{\gtrsim} = \prod_{i=1}^{n} \overset{l}{\gtrsim}_i$, where each player i ($i = 1, \cdots, n$) has preorderings $\underset{c}{\gtrsim}_i$ and $\underset{l}{\gtrsim}_i$ which are independent of each others' ones.

Now, we must decide *collective rationality postulates* as like Conditions 1, 2, and 3 which are the individual common rationality postulates. Therefore, let the collective rationality postulates also be Conditions 1, 2 and 3 which replace the one-dimentions' metric spaces Y and $M(Y)$ with the product metric spaces Y^n and $M^n(Y)$. Then we have a multivariete von Neumann-Morgenstern utility as follows;

Theorem 2. *Let Y be a separable metric space. Let $\overset{c}{\gtrsim}$ be a complete preordering defined on the product metric space Y^n, and $\overset{l}{\gtrsim}$ a complete preorder-*

ing defined on σ–convex subspace P of $M^n(Y)$ such that $D \subset P$. Then, Conditions 1, 2 and 3 are necessary and sufficient for the existence of a real–valued continuous bounded function h defined on P by $h(p) = \int_{Y^n} u\, d(\prod_{i=1}^{n} p_i)$ for any $p = (p_1, \cdots, p_n) \in P$, where p_i is a product probability measure on $B^n(Y)$, is order–preserving with respect to $\overset{l}{\gtrsim}$ and σ–linear. Any h and u are unique up to a positive linear transformation. (For proof, see Nishimura, K., [5] .)

According to Theorem 2, we have gotten a *collective expected utility h*. Next, we shall consider how relation between the collective expected utility and each player's expected utility there exists. We shall introduce the preorderings induced by the preordering $\overset{l}{\gtrsim}$ which is defined on a product subspace P of $M^n(Y)$. We assume that each player has *references* of lottery tickets which denote the origin and the unit interval. Let a n–tuple of any origin's lottery ticket r_i which each player i decides and fixes $(r_1, \cdots, r_n)$. Then we define a preordering $\overset{l}{\gtrsim}_i$ on $Y (i = 1, \cdots, n)$ induced by $\overset{l}{\gtrsim}$ on Y^n, as follows;

$x_i \overset{l}{\gtrsim}_i y_i$ means $(r_1, \cdots, r_{i-1}, x_i, r_{i+1}, \cdots, r_n) \overset{l}{\gtrsim} (r_1, \cdots, r_{i-1}, y_i, r_{i+1}, \cdots, r_n)$, for any $(r_1, \cdots, r_{i-1}, x_i, r_{i+1}, \cdots, r_n)$ and $(r_1, \cdots, r_{i-1}, y_i, r_{i+1}, \cdots, r_n)$ in Y^n.

Then we have an additive form of the collective expected utility as follows;

Theorem 3. *Under the assumptions of Theorem 2, a linear and continuous collective expected utility h defined on P has the form $h(p) = \sum_{i=1}^{n} h_i(p_i)$ for any $p = (p_1, \cdots, p_n) \in P$, where each $h_i(p_i)$ is linear, continuous and order–preserving with respect with the induced preordering $\overset{l}{\gtrsim}_i$. This form is unique up to a positive linear transformation. (For proof, see Nishimura, K.,* [5] .)

Using this Theorem 3, we may show the correlated relation between individual preferences and collective preferences which Harsanyi, *J. C.* has understood as the personal preferences and the moral preferences [3] . Since each player's expected utility $h_i(p_i)$ is defined by $h_i(p_i) = h(r_1^2, \cdots, r_{i-1}^2, p_i, r_{i+1}^2, \cdots, r_n^2)$ where $r^2 = (r_1^2, \cdots, r_n^2)$ satisfies $h(r^2) = 0$, each player fixes his reference element r_i^2 until the game is over.

On the other hand, according to Theorem 1 each player cannot know the others' fixed references. The collective preferences, however, are defined by the product preferences which are independent of each other's individual preferences called the induced preordering $\overset{l}{\underset{\sim}{>}}_i$. Therefore, we shall consider that the individual expected utility given by Theorem 1 is equivalent to his induced expected utility given by Theorem 3 though their relation is tied with a positive linear transformation. Thus, we can proceed to the above task to show the correlated relation between the individual preferences and the collective preferences by the framework of Theorem 3.

Let a player i $(i = 1, \cdots, n)$ select any *subjective* other players' fixed references $e^i_1, \cdots, e^i_{i-1}, e^i_{i+1}, \cdots, e^i_n$ and his own fixed reference r^2_i. Then, by Theorem 3, his induced expected utility function $h^*_i(p_i)$ can be obtained, since the product preordering $\overset{l}{\underset{\sim}{>}}$ is defined such that each induced preordering has no effect on the others' ones. Since any player can change the others' fixed references, we may interpret that he has *subjective collective preferences* which are represented by $h^*_i(p_i) = h^{*^i}(e^i_1, \cdots, e^i_{i-1}, p_i, e^i_{i+1}, \cdots, e^i_n)$.

Next, we shall show the relation between the collective induced expected utility function $h_i(p_i)$ and the subjective one $h^*_i(p_i)$, using the results in [6, the transformation formula (4)] , as follows;

$$h_i(p_i) = \frac{1}{h^{*^i}(r^1) - h^{*^i}(r^2)} \left(h^*_i(p_i) - h^*_i(r^2_i)\right) . \qquad (1)$$

Since $h^*_i(r^2_i) = 0$ $(i = 1, \cdots, n)$, using Theorem 3 we have the following linear form,

$$h(p) = \sum_{i=1}^{n} h_i(p_i) = \sum_{i=1}^{n} \frac{1}{h^{*^i}(r^1) - h^{*^i}(r^2)} h^*_i(p_i) . \qquad (2)$$

This result shows that by using the multivariate von Neumann–Morgenstern utility the problem which Harsanyi, J. C. has considered in his papers appears to be solved as the above. The left problems in our formulation are whether the form (2) contains the interpersonal utility comparisons and the product preordering $\overset{l}{\underset{\sim}{>}}$ is defined such that it is a Pareto preordering that if $p', p'' \in P$ and for all $i = 1, \cdots, n$, $p' \overset{l}{\underset{\sim}{>}}_i p''$, then $(p', \cdots, p') \overset{l}{\underset{\sim}{>}} (p'', \cdots, p'')$. They might be affirmatively solved.

References

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