ON THE LORENTZ INVARIANCE OF MAXWELL' S EQUATIONS

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§0. Maxwell' s equations in vacuum take the form

$$\partial_t E - \operatorname{curl} H = \partial_t H + \operatorname{curl} E = 0$$
$$\operatorname{div} E = \operatorname{div} H = 0$$

setting the speed of light $c = 1$. Here E denotes the electric field and H the magnetic field. Setting $F = E + iH$ $(i^2 = -1)$, we have

$$\partial_t F + i \operatorname{curl} F = 0 \qquad (0.1)$$
$$\operatorname{div} F = 0.$$

Let us difine a matrix by

$$T = (T^{ij}) = \begin{pmatrix} 0 & -iF_3 & iF_2 & -F_1 \\ iF_3 & 0 & -iF_1 & -F_2 \\ -iF_2 & iF_1 & 0 & -F_3 \\ F_1 & F_2 & F_3 & 0 \end{pmatrix}, \quad F = {}^t(F_1, F_2, F_3).$$

Then we have the equations in a covariant form

$$\sum_{j=1}^{4} \partial_j T^{ij} = 0 \qquad (1 \leq i \leq 4, \ x_4 = t) \qquad (0.2)$$

(see [5]) . We regard T^{ij} as a (2,0)-tensor field in R^4.
Namely we set

$$\overline{T}^{ij} = \sum_{k,l} B_{ik} B_{jl} T^{kl}, \quad B = (B_{ij}) = A^{-1}, \qquad (0.3)$$

for another coordinates $\bar{x}$ relating to x as $x = A\bar{x} + a$ with a non-singular matrix A and $a \varepsilon R^4$. We can see easily that the system (0.2) is written with the new coordinates as

$$\sum_{j=1}^{4} \bar{\partial}_j \overline{T}^{ij} = 0. \qquad (0.4)$$

However under these affine transformations T^{ij} does not preserve its form, that is : $\overline{T}^{ij}$ does not come from a single field $\overline{F}$. We say that a non-singular matrix A is a conformal Lorentz transformation, if A satisfies

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$$ {}^{t}AJA = \lambda J \qquad \text{with} \quad \lambda > 0, \qquad J = \begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & -1 \end{pmatrix} $$

In this note we shall show that if A is a proper conformal Lorentz transformation (see §1 for the definition), there exists a representation $\hat{B}(A)$ of A such that T comes from the field $F = \hat{B}(A)\overline{F}$. Hence (0.4) is written as

$$ (0.5) \qquad \bar{\partial}_t \overline{F} + i\,\overline{\mathrm{curl}}\ \overline{F} = 0, \qquad \overline{\mathrm{div}}\ \overline{F} = 0. $$

Let us return to the covariant form (0.2) of the equations. Let X be an infinitesimal conformal Lorentz transformation. We shall see that derivatives $\partial_j (1 \leq j \leq 4)$ commute with the Lie derivative L_X, that is:

$$ (0.6) \qquad L_X \sum_{j=1}^{4} \partial_j T^{ij} = \sum_{j=1}^{4} \partial_j (L_X T)^{ij} = 0. $$

Moreover we shall show that there exists a vector field $\hat{X}$ such that $L_X T$ also comes from the field $\hat{X}F$. Consequently we find by (0.6) that $\hat{X}F$ satisfies

$$ (0.7) \qquad \partial_t(\hat{X}F) + i\mathrm{curl}(\hat{X}F) = 0, \qquad \mathrm{div}(\hat{X}F) = 0. $$

As is well known, the energy integral

$$ \| F(t) \|^2 = \int_{R^3} | F(x, t) |^2 dx $$

does not vary with t, if F is a solution to Maxwell' s equations. (0.7) says that $\| \hat{X}F(t) \|$ and even $\| \hat{X}^n F(t) \|$ are constant with respect to t. Thus we find that $\| \hat{X}^n F(t) \|$ is estimated by the derivatives of $F(x, 0)$.

In conclusion, we should reffer to the works of S. Klainerman ([2], [3]) on non–linear wave equations and non–linear Klein–Gordon equations. For the D' Alambertian $\Box$, we can show directly

$$ [\Box, X] = 0 \qquad mod\ (\Box), $$

if X is an infinitesimal conformal Lorentz transformation. Thus we find that u is a solution to the wave equation

$$ \Box u = 0, $$

then the energy of $X^n u$ is estimated with the derivatives of the initial value of u. Klainerman has effectively used these arguments for establishing global in time solutions to those equations. Following the same way as Klainerman, we can obtain global solutions to non–linear Maxwell' s equations.

§1. A point x in R^{n+1} is represented by an ordered n-tuple of real numbers

$(x_1, x_2, \cdots, x_{n+1})$, which is called the canonical coordinates. R^{n+1} is endowed with the Lorentz metric

$$\sum_{j=1}^{n} x_j y_j - x_{n+1} y_{n+1} = (x, Jy) \qquad J = \begin{pmatrix} I_n & 0 \\ 0 & -1 \end{pmatrix}$$

The matrices satisfying

$$ {}^t AJA = \lambda J \ (\lambda > 0)$$

constitute an $n(n+1)/2+1$ dimensional Lie group, which is called the conformal Lorentz group and denoted by $CO(n, 1)$. The connected component of E (the unit matrix), denoted by $CSO^+(n, 1)$ is represented as

$$CSO^+(n, 1) = \{A = (a_{ij}) \in M(n+1) \mid {}^t AJA = \lambda J \ (\lambda > 0), \ \det A > 0, \ a_{n+1\, n+1} > 0\}.$$

We call $CSO^+(n, 1)$ the proper conformal Lorentz group. We say that $\bar{x} = (\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_{n+1})$ is an admissible system of coordinates of R^{n+1} when $\bar{x}$ is related to the canonical coordinates x as

$$x = A\bar{x} + a, \ A \in CSO^+(n,1), \ a \in R^{n+1}.$$

We denote R^{n+1} with the admissible coordinates by M^{n+1}, which we call the Minkowski space. Set

$$co(n, 1) = \{\alpha = (\alpha_{ij}) \in M(n+1) \mid \alpha = \alpha' + \lambda I, \ {}^t\alpha' J + J\alpha' = 0, \ \lambda \in R\}.$$

Then $co(n, 1)$ is the Lie algebra associated to $CO(n,1)$. A vector field expressed as

$$X = \sum_{i,j=1}^{n+1} (\alpha_{ij} x_j + \alpha_j)(\partial_i), \ (\alpha_{ij}) \in co(n, 1), \ (\alpha_j) \in R^{n+1}.$$

is called an infinitesimal (inhomogeneous) conformal Lorentz transformation (see [1] and [4]). We denote by $L(M^{n+1})$ the set of the infinitesimal conformal Lorentz transformations.

Let T be a tensor field of type (r, s) defined on M^{n+1}. T is regarded as an $(n+1)^{r+s}$-tuple of functions $T^{i_1 \cdots i_r}_{i_1 \cdots i_j}(x)$ which are subject to the rule of transformation

$$T^{i_1 \cdots i_r}_{i_1 \cdots i_j}(x) = \sum_{k,l} A_{i_1 k_1} \cdots A_{i_r k_r} B_{l_1 j_1} \cdots B_{l_s j_s} \bar{T}^{k_1 \cdots k_r}_{l_1 \cdots l_s}(\bar{x})$$

for $x = A\bar{x} + a$, $A = (A_{ij}) \in CSO^+(n,1)$, $B = (B_{ij}) = A^{-1}$. We can easily verify if $T \in T^{(r,s)}$, then

$$\partial_{j_{s+1}} T^{i_1 \cdots i_r}_{j_1 \cdots j_s} \in T^{(r, s+1)}.$$

Namely, ∂_j works as a covariant derivative on M^{n+1}.

Let $X = \sum_j \xi^j \partial_j$, $Y = \sum_j \eta^j \partial_j$ be vector fields on M^{n+1}. The Lie derivative of Y in X is defined by

$$L_X Y = \sum_{j,k} (\xi^k \partial_k \eta^j - \eta^k \partial_k \xi^j)\partial_j.$$

For a 1-form ω on M^{n+1}, we define in a similar manner

$$L_X \omega = \sum_{j,k} (\xi^k \partial_k \omega_j + \partial_j \xi^k \omega_k) dx^j.$$

The Lie derivatives can be extended in a unique way for the (r, s)–tensors (see[1] and [4]). Especially for $X \in L(M^{n+1})$, we have

Proposition 1. 1. *Let $X = \sum_{i,j} (\alpha_{ij} x_j + \alpha_i)\partial_i$ be an in-homogeneous infinitesimal conformal Lorentz transformation. For $Y \in T^{(1,0)}$ and $\omega \in T^{(0,1)}$, Lie derivatives are expressed as*

$$L_X Y = \sum_{j,k,l} (\alpha_{kl} x_l \partial_k \eta^j - \alpha_{jk} \eta^k)\partial_j$$

$$L_X \omega = \sum_{j,k,l} (\alpha_{kl} x_l \partial_k \omega_j + \omega_k \alpha_{kj}) dx^j.$$

The consideration in this note is based on the next theorem.

Theorem 1.2. *If $X \in L(M^{n+1})$, L_X commutes with ∂_j that is:*

$$[L_X, \partial_j] = 0 \quad (j = 1, 2, \ldots, n+1).$$

Proof. We prove the theorem for $S = (S^{ij}) \in T^{(2,0)}$.
Setting $X = \sum_{i,j} (\alpha_{ij} x_j + \alpha_i)\partial_i$, we have

$$(L_X S)^{ij} = \sum_{l,m} (\alpha_{ml} x_l + \alpha_m)\partial_m S^{ij} - \alpha_{jm} S^{im}.$$

Hence

$$\begin{aligned}\partial_k (L_X S)^{ij} &= \sum_{l,m} (\alpha_{ml} x_l + \alpha_m)\partial_m \partial_k S^{ij} \\ &\quad - \alpha_{im} \partial_k S^{mj} - \alpha_{jm} \partial_k S^{im} + \partial_m S^{ij} \alpha_{mk} \\ &= (L_X \partial_k S)^{ij}.\end{aligned}$$

§2. We return to Maxwell's equations in the form

(2.1) $$\partial_t F + i \operatorname{curl} F = 0$$
$$\operatorname{div} F = 0 \qquad (i^2 = -1, F = E + iH).$$

For $F = {}^t(F_1, F_2, F_3)$, we define $\Omega(F)$ by

$$\Omega(F) = \begin{pmatrix} 0 & -F_3 & F_2 \\ F_3 & 0 & -F_1 \\ -F_2 & F_1 & 0 \end{pmatrix}.$$

Using an anti–symmetric matrix T defined by

(2.2) $$T = (T^{ij}) = \begin{pmatrix} i\,\Omega(F) & -F \\ {}^tF & 0 \end{pmatrix}$$

and denoting $\partial_4 = \partial_t$, we have the equations in a covariant form

(2.3) $$\sum_j \partial_j T^{ij} = 0 \qquad (1 \le i \le 4, x_4 = t).$$

We regard T^{ij} as a (2, 0)-tensor in M^{3+1}. For another admissible coordinates $\bar{x}$ which is related to x as

$$x = A\bar{x} + a \qquad \text{with } A = (A_{ij}) \in CSO^+(3,1),\ a \in R^4,$$

we set

(2.4) $$\bar{T}^{ij}(\bar{x}) = \sum_{k,l} B_{ik} B_{jl} T^{kl}(x)$$
$$\bar{\partial}_j = \sum_k A_{kj} \partial_k \qquad \text{with} \quad B = (B_{ij}) = A^{-1}.$$

Then the equations are written with the new coordinates as

(2,3) $$\sum_j \bar{\partial}_j \bar{T}^{ij} = 0 \qquad (1 \le i \le 4).$$

We show that $\bar{T}$ has the same form as (2.2).

Lemma 2. 1. *Let $\bar{T}$ be the tensor field defined by* (2. 4) *with $A \in CSO^+$* (3, 1). *Then there exists a representation $\hat{B}(A)$ of A such that T is expressed as*

(2.5) $$\bar{T} = (\bar{T}^{ij}) = \begin{pmatrix} i\Omega(\bar{F}) & -\bar{F} \\ {}^t\bar{F} & 0 \end{pmatrix} \qquad \textit{with} \qquad \bar{F} = \hat{B}(A)F.$$

Proof. We observe that

$$\bar{T} = BT{}^tB.$$

Since $CSO^+(3,1)$ is generated by the following subgroups

$$G_1 = \left\{ U \oplus 1 = \begin{pmatrix} U & 0 \\ 0 & 1 \end{pmatrix} \;\middle|\; U \in SO(3) \right\}$$

$$G_2 = \left\{ L(\theta) = \begin{pmatrix} 1 & 0 & & \\ 0 & 1 & & 0 \\ & & \cosh\theta & \sinh\theta \\ & 0 & \sinh\theta & \cosh\theta \end{pmatrix} \;\middle|\; \theta \varepsilon R \right\}$$

$$G_3 = \{\lambda I \mid \lambda > 0\},$$

it is sufficient to prove the lemma for each subgroup.

(i) For $A = U^{-1} \oplus 1 \in G_1$, we have

$$BT{}^tB = \begin{pmatrix} i\,U\Omega(F)U & -UF \\ {}^t(UF) & 0 \end{pmatrix}.$$

we shall show

$$(2.6)\qquad {}^tU\Omega(F)U = \Omega(UF).$$

Thus we find that (2. 5) holds with $\hat{B}$ $(A) = U$. Note that

$$\Omega(F)u = F \times u \qquad u \in R^3$$

where $F \times u$ denotes the exterior product of F and u. Then it follows that (2. 6) is equivalent to

$${}^tU(F \times (Uv)) = (UF) \times v.$$

Denoting by U_j the j–th column vector of U, we find that U_j satisfies

$$(2.7)\qquad \begin{aligned} &(U_i,\ U_j) = \delta_{ij} \\ &U_2 \times U_3 = U_1\,,\ U_3 \times U_1 = U_2\,,\ U_1 \times U_2 = U_3\,. \end{aligned}$$

For $v = {}^t(v_1, v_2, v_3)$, we have

$$Uv = v_1 U_1 + v_2 U_2 + v_3 U_3\,.$$

Hence

$$\begin{aligned}
&{}^tU(F\times Uv) \\
= &\quad v_1\begin{pmatrix}(U_1,\ F\times U_1)\\(U_2,\ F\times U_1)\\(U_3,\ F\times U_1)\end{pmatrix} + v_2\begin{pmatrix}(U_1,\ F\times U_2)\\(U_2,\ F\times U_2)\\(U_3,\ F\times U_2)\end{pmatrix} + v_3\begin{pmatrix}(U_1,\ F\times U_3)\\(U_2,\ F\times U_3)\\(U_3,\ F\times U_3)\end{pmatrix} \\
= &-v_1\begin{pmatrix}0\\(F,\ U_2\times U_1)\\(F,\ U_3\times U_1)\end{pmatrix} - v_2\begin{pmatrix}(F,\ U_1\times U_2)\\0\\(F,\ U_3\times U_2)\end{pmatrix} - v_3\begin{pmatrix}(F,\ U_1\times U_3)\\(F,\ U_2\times U_3)\\0\end{pmatrix} \\
= &\quad \begin{pmatrix}(U_2,F)\ v_3 - (U_3,F)\ v_2\\(U_3,F)\ v_1 - (U_1,F)\ v_3\\(U_1,F)\ v_2 - (U_2,F)\ v_1\end{pmatrix} \\
= &\quad {}^t(UF)\ \times v\,.
\end{aligned}$$

Thus (2.6) is proved.

(ii) For $A = L(\theta) \in G_2$, direct computations show that
$BT{}^tB = L(-\theta)TL(-\theta)$

$$
=\begin{pmatrix}
0 & -iF_3 & iF_2\cosh\theta+F_1\sinh\theta & -iF_2\sinh\theta-F_1\cosh\theta \\
iF_3 & 0 & -iF_1\cosh\theta+F_2\sinh\theta & iF_1\sinh\theta-F_2\cosh\theta \\
-iF_2\cosh\theta-F_1\sinh\theta & iF_1\cosh\theta-F_2\sinh\theta & 0 & -F_3 \\
iF_2\sinh\theta+F_1\cosh\theta & -iF_1\sinh\theta+F_2\cosh\theta & F_3 & 0
\end{pmatrix}.
$$

Then setting

$$
\begin{pmatrix}\bar{F}_1\\ \bar{F}_2\\ \bar{F}_3\end{pmatrix}=\begin{pmatrix}\cosh\theta & i\sinh\theta & 0\\ -i\sinh\theta & \cosh\theta & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}F_1\\ F_2\\ F_3\end{pmatrix}
$$

$$
=\hat{B}(A)F,
$$

we have

$$
BT{}^{t}B=\begin{pmatrix} i\Omega\ (\bar{F}) & -\bar{F}\\ {}^{t}\bar{F} & 0\end{pmatrix}.
$$

(iii) For $A=\lambda I\in G_3$, we have

$$
BT{}^{t}B=\lambda^{-2}T.
$$

Thus we can see that (2.6) holds clearly by setting $\bar{F}=\lambda^{-2}F$. In this way we have proved the lemma.

By Lemma 2.1 we have immediately

Theorem 2. 2. *Let $\bar{x}$ be an admissible coordinates relating to x as $x = A\bar{x} + a$ with $A \in CSO^{+}(3,1)$. Then there exists a representation $\hat{B}(A)$* of *A such that Maxwell's equations are expressed with the new coordinates as*

$$
\begin{aligned}
\bar{\partial}_{\bar{t}}\bar{F} + i\,\overline{\mathrm{curl}}\bar{F} &= 0\\
\overline{\mathrm{div}}\bar{F} &= 0 \qquad (\bar{t}=\bar{x}_4)
\end{aligned}
$$

with $\bar{F}=\hat{B}(A)F.$

§3. In this section we seek certain conservation laws for Maxwell's equations by using Theorem 1.2 and Theorem 2.2. Theorem 1.2 says that if T is a solution to the equations, then $L_X T$ is also a solution with $X \in L(M^{3+1})$. For

$$X = \sum_{i,j} (a_{ij}x_j + a_i)\partial_i$$

$L_X T$ is expressed as

$$L_X T = XT - aT - T{}^t a. \tag{3.1}$$

Let $\{A(s)\}$ be a one parameter family in $CSO^+(3,1)$ such that $A(0) = I$ and $A'(0) = a$. It follows from Theorem 2.2 that

$$\bar{T}(\bar{x}) = B(s)T(A\bar{x} + a){}^t B(s) = \begin{pmatrix} i\Omega(\hat{B}(s)F) & -\hat{B}(s) \\ {}^t(\hat{B}(s)F) & 0 \end{pmatrix} \tag{3.2}$$

$$(B(s) = A(s)^{-1}).$$

Differentiating (3.2) in s at $s = 0$, we have

$$XT - aT - T{}^t a = \begin{pmatrix} i\Omega((\hat{X}+\beta)\ F) & -\ (\hat{X}+\beta)F \\ {}^t((\hat{X}+\beta)\ F) & 0 \end{pmatrix} \qquad (\hat{\beta} = \hat{B}'(0)).$$

Thus we have proved

Proposition 3.1. *For $X \in L(M^{3-1})$, there exists a vector field $\hat{X} = X + \hat{\beta}$ ($\hat{\beta} \in M(3)$) such that $L_X T$ is expressed as*

$$L_X T = \begin{pmatrix} i\Omega(\hat{X}F) & -\hat{X}F \\ {}^t(\hat{X}F) & 0 \end{pmatrix}. \tag{3.3}$$

We know that $L(M^{3+1})$ is generated by

$$R_1 = x_2\partial_3 - x_3\partial_2,\ R_2 = x_3\partial_1 - x_1\partial_3,\ R_3 = x_1\partial_2 - x_2\partial_1$$

$$L_j = x_4\partial_j + x_j\partial_4 \ \ (1 \le j \le 3)$$

$$\partial_j \ \ (1 \le j \le 4)$$

$$L_0 = x_1\partial_1 + x_2\partial_2 + x_3\partial_3 + x_4\partial_4.$$

Denoting

$$A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix},\quad A_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix},\quad A_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

we have

Proposition 3.2. *$\hat{R}_j$, $\hat{L}_j$ and $\hat{\partial}_j$ are expressed as*

$$\hat{R}_j = R_j - A_j \qquad (1 \leq j \leq 3)$$
$$\hat{L}_j = L_j - iA_j \qquad (1 \leq j \leq 3)$$
$$\hat{\partial}_j = \partial_j \qquad (1 \leq j \leq 4)$$
$$\hat{L}_0 = L_0 .$$

Theorem 1.2 says that if F is a solution to the Maxwell equations, then $\hat{X}F$ is also a solution to the equations. Finally we compute the commutator of the Maxwell operator and these vector fields.

Theorem 3.3 *Set $L = \partial_t + i\mathrm{curl}$. Then it follows*

(i) $[L, \hat{R}_j] = 0$, $\mathrm{div}\hat{R}_j - R_j\mathrm{div} = 0$

(ii) $[L, \hat{L}_j] = iA_j L + e_j \otimes \mathrm{div}$, $\mathrm{div}\hat{L}_j - L_j\mathrm{div} = e_j \bullet L$

(iii) $[L, \hat{\partial}_j] = 0$, $\mathrm{div}\hat{\partial}_j - \partial_j\mathrm{div} = 0$

(iv) $[L, \hat{L}_0] = L$, $\mathrm{div}\hat{L}_0 - L_0\mathrm{div} = \mathrm{div}$

where $e_j = {}^t(0, \ \overset{j}{\check{1}}, 0)$.

Proof. The Maxwell operator is expressed as

$$\partial_t + i\mathrm{curl} = \partial_t + i \sum_{j=1}^{3} A_j \partial_j .$$

One can easily check the relations by direct computation using

$$[A_2, A_3] = A_1, \quad [A_3, A_1] = A_2, \quad [A_1, A_2] = A_3 .$$

Added in proof. By the proof of Lemma 2. 1, *we can see that $\hat{B}(A)$ belongs to the conformal complex rotation group $CSO(3, \boldsymbol{C}) = \{\hat{B} \in M(3, \boldsymbol{C}) \mid {}^t\hat{B}\hat{B} = \lambda I$ $(\lambda > 0)$, $\det \hat{B} > 0\}$. Since $CSO(3, \boldsymbol{C})$ is generated by $\{\hat{B}(A) \mid A = U \oplus 1 \in G_1$, $A = L(\theta) \in G_2$, $A = \lambda I \in G_3\}$, we find that the correspondence $A \to \hat{B}(A)$ gives an isomorphism between $CSO^+(3, 1)$ and $CSO(3, \boldsymbol{C})$.*

References

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