A REMAINDER ESTIMATE FOR THE ASYMPTOTIC LAW FOR THE DISTRIBUTION OF EIGENVALUES OF THE LAPLACIAN IN A POLYGONAL DOMAIN

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§ 0. Let D be a bounded domain in R^2. We consider the asymptotic distribution of eigenvalues of the Laplacian with the Dirichlet con dition

$$(0.1)\qquad \begin{cases} \Delta u + \lambda u = 0 & \text{in } D \\ u = 0 & \text{on } \partial D. \end{cases}$$

Let $\{\lambda_n\}$ denote the set of the eigenvalues and $N(\nu)$ denote the number of eigenvalues satisfying $\sqrt{\lambda_n} \leqslant \nu$. 1911, H. Weyl proved

$$N(\nu) = \frac{|D|}{4\pi}\nu^2 + o(\nu^2) \qquad \text{as } \nu \to \infty,$$

where D denotes the area of D. Later Weyl and then R. Courant obtained a better remainder estimate $O(\nu\log\nu)$. A proof is found in Courant-Hilbert [3].

In this note we shall show that we can replace the remainder esti mate with $O(\nu)$, if D is a polygonal domain. This fact was firstly proved by P.B. Bailey and F.H. Brownell in [1]. They employed the Green function of the heat equation and obtained the asymptotic trace formula

$$\sum_{j=1}^{\infty} e^{-\lambda_n t} = \frac{|D|}{4\pi t} - \frac{|\partial D|}{8\sqrt{\pi t}} + \beta + O(e^{-\varepsilon/t^2}) \text{ as } t \to 0.$$

Then together with Ganelius' Tauberian theorem, they obtained the result. In this paper we shall make use of the Green function of the wave equation instead and obtain the result merely by computing the Fourier transform of the trace of the Green function. In section 1, we shall give an asymptotic representation of $N(\nu)$ using the

Green function of the wave equation. In section 2, we shall give a representation of the Green function for small time by patching up the Green functions of sectors. Then in section 3 and 4 we shall obtain the estimate.

We should note that R. Seeley and T.L. Pham obtained, in 1978, the remainder estimate $O(\nu)$ for the domain with sufficiently smooth boundary and finally V. Ivrii proved the following asymptotic formula, which is known as the Weyl conjecture, for the Laplacian in a Riemannian manifold with a smooth boundary under some reasonable additional conditions. In our case the formula reads

$$N(\nu) = \frac{|D|}{4\pi}\nu^2 - \frac{|\partial D|}{4\pi}\nu + o(\nu) \text{ as } t \to \infty.$$

As for irregular domains, P. Bérard, in [2], proved the Weyl conjecture for the domains which can be considered the fundamental regions of affine Weyl groups.

I would like to express my hearty thanks to professor K. Ibuki of Kobe University of Commerce who made comperhensive expositions of classical methods of the construction of Green functions of the wave equation. I owe entrely to him all the knowledge of these matters.

§ 1. In this section we shall give a representation of $N(\nu)$ by means of the following solution of the wave equation. We just follow the argument of Bérard [2].

$$(1.1)\quad \begin{cases} (\frac{\partial^2}{\partial t^2} - \Delta)e(x,\xi,t) = 0 & x,\xi \in D, t \in R \\ e(x,\xi,0) = \delta(x-\xi) & x,\xi \in D \\ \frac{\partial}{\partial t}e(x,\xi,0) = 0 & x,\xi, \in D \\ e(x,\xi,t) = 0 & y \in \partial D, \xi \in D, t \in R. \end{cases}$$

Let $\{\varphi_j\}$ be an orthonormal system of the eignfunctions of the Laplacian. Then $e(x,\xi,t)$ is expressed as

$$e(x,\xi,t) = \sum_{j=1}^{\infty} \cos\mu_j t \, \varphi_j(x)\varphi_j(\xi),$$

where $\mu_j = \sqrt{\lambda_j}$ and the infinite sum converges in appropriate distribuiton spaces. $e(x, \xi, t)$ is considered to be a distribution in t for every (x,ξ). Then

$$(1.2) \quad <e(x,\xi,t),\ \phi(t)> = \sum_{j=1}^{\infty} \varphi_j(x)\varphi_j(\xi) \int_{-\infty}^{\infty} \phi(t)\cos\mu_j t\ dt$$

converges in the usual sence and both sides are understood to be smooth functions in (x, ξ). Here $<e(x, \xi, t), \phi(t)>$ denotes that $e(x, \xi, t)$ is considered to be a distribution t. We insert$x = \xi$ and by integrating both sidesof (1.2) x, we have

Proposition 1.1.

$$(1.3) \quad \int_D <e(x,x,t), \phi(t)>dx = \sum_{j=1}^{\infty} \int_{-\infty}^{\infty} \phi(t)\cos\mu_j t\ dt.$$

Hereafter, the next auxiliary function will be frequently used.

Lemma 1.2. *There exists a smooth, positive and even function* $\rho(\mu)$ *such that* $\rho(t)$ *(the Fourier transform of* ρ*) is positive, even,* $\rho(0)=1$, supp $\rho \subset [-T,T]$ *and decreasing in* $t>0$.

Proof. We choose a smooth function $\zeta(t)$ which is positive, even, $\int_{-\infty}^{\infty}\zeta(t)^2 dt = 1$, supp $\zeta \subset [-1/2, 1/2]$. We set $\rho_1(t)$

$$= \int_{-\infty}^{\infty} \zeta(t-s)\zeta(s)\ ds$$

Then $\rho(t) = \rho_1(T^{-1}t)$ has the desired properties.

We insert $\psi(t) = \frac{1}{2\pi} e^{it\mu}\rho(t)$ in (1.3). When we set

$$P(\mu) = \frac{1}{2\pi}\int_D <e(x,x,t),\ e^{-it\mu}\rho(t)>dx$$

we obtain

Proposition 1.3.

$$(1.4) \quad P(\mu) = \frac{1}{2}\sum_{j=1}^{\infty} [\rho(\mu+\mu_j) + \rho(\mu-\mu_j)].$$

Proof. $P(\mu) = \sum_{j=1}^{\infty} \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{it\mu}\ \rho(t)\ \cos\mu_j t\ dt$

$$= \frac{1}{2}\sum_{j=1}^{\infty}\frac{1}{2\pi}\int_{-\infty}^{\infty} [e^{it(\mu+\mu_j)} + e^{it(\mu-\mu_j)}]\ \rho(t)\ dt.$$

Then we have the proposition. In addition, we can see easily that $P(\mu)$ is an even function in μ.

On the other hand. $N(\nu)$ is represented as

$$N(\nu)=\sum_{\mu_j\leqslant\nu}1$$
$$=\sum_{\mu_j\leqslant\nu}\int_{-\infty}^{\infty}\rho(\mu-\mu_j)\,d\mu$$
$$=\frac{1}{2}\sum_{\mu_j\leqslant\nu}\int_{-\infty}^{\infty}\rho(\mu+\mu_j)+\rho(\mu-\mu_j)\,d\mu.$$

When we set

$$R_1(\nu)=\sum_{\mu_j\leqslant\nu}\int_{-\infty}^{-\nu}\rho(\mu-\mu_j)\,d\mu$$
$$R_2(\nu)=\sum_{\mu_j\leqslant\nu}\int_{\nu}^{\infty}\rho(\mu-\mu_j)\,d\mu$$
$$R_3(\mu)=\sum_{\mu_j>\nu}\int_{-\nu}^{\nu}\rho(\mu-\mu_j)\,d\mu,$$ we have

Proposition 1.4.

$$N(\nu)=\int_{-\nu}^{\nu}P(\mu)\,d\mu+\sum_{j=1}^{3}R_j(\nu) \tag{1.5}$$

Next we shall show that each $R_j(\nu)$ behaves like $O(\nu)$ as $\nu\to\infty$.

Lemma 1.5. *When μ tends to ∞, we find*
(1.6) $\sharp\{\mu_j \mid |\mu-\mu_j|\leqslant a\}\leqslant C(a)\,P(\mu)$, *where $C(a)$ is a positive constant depending only on a.*

Proof. Since $P(\mu)\geqslant\frac{1}{2}\sum_{j=1}^{\infty}\rho(\mu-\mu_j)$

$$\geqslant\frac{1}{2}\times{}^{\sharp}\{\mu_j \mid |\mu-\mu_j|\leqslant a\}\times\inf\{\rho(\mu)\mid |\mu|\leqslant a\},$$

we find $\sharp\{\mu_j \mid |\mu-\mu_j|\leqslant a\}\leqslant\frac{2}{\inf\{\rho(\mu)\mid |\mu|\leqslant a\}}P(\mu)$.

Proposition 1.6. *If we assume that $P(\mu)=O(\mu)$ holds as μ tends to ∞, then we have*

(1) $R_1(\nu)=O(\nu^{-N})$ $(N>0)$, (2) $R_2(\nu)=O(\nu)$, (3) $R_3(\nu)=O(\nu)$ *as ν tends to ∞.*

Proof. (1) $R_1(\nu) = \sum_{\nu\leqslant\mu_j}\int_{-\infty}^{-\nu-\mu_j}\rho(\mu)d\mu$

$$\leqslant N(\nu)\int_{-\infty}^{-\nu}\rho(\mu)d\mu.$$

Since $N(\nu)=O(\nu^2)$, $\rho(\mu)=O(\mu^{-\infty})$, we fiind $R_1(\nu)=O(\nu^{2-N})$ for every N as $\nu\to\infty$.

(2) $R_2(\nu) = \sum_{\mu_j\leqslant\nu}\int_{\nu-\mu_j}^{\infty}\rho(\mu)\,d\mu$

$$= \sum_{k=0}^{\infty}\sum_{k\leqslant\nu-\mu_j<(k+1)}\int_{\nu-\mu_j}^{\infty}\rho(\mu)\,d\mu.$$

$$\leqslant \sum_{k=0}^{\infty}{}^{\#}\{\mu_j|\ |\nu-\mu_j-k|\leqslant 1\}\int_k^{\infty}\rho(\mu)\,d\mu.$$

By Proposition 1.5, we find

$$R_2(\nu)\leqslant C\sum_{k=0}^{\infty}P(\nu-k)\int_k^{\infty}\rho(\mu)\,d\mu.$$

Since we assumed $P(\mu)\leq U\ |\mu|$ we have

$$R_2(\nu)\leqslant C\sum_{k=0}^{\infty}(\nu+k)\int_k^{\infty}\rho(\mu)\,d\mu$$

$$\leqslant C\sum_{k=0}^{\infty}\frac{(\nu+k)}{(1+k)^N}\leqslant C\nu \quad \text{as } \nu\to\infty.$$

(3) $R_3(\nu) = \sum_{\mu_j>\nu}\int_{-\nu-\mu_j}^{\nu-\mu_j}\rho(\mu)\,d\mu$

$$+\sum_{\mu_j<\nu}\int_{-\infty}^{\nu-\mu_j}\rho(\mu)\,d\mu$$

$$=\sum_{k=0}^{\infty}\sum_{-(k+1)<\nu-\mu_j\leqslant -k}\int_{-\infty}^{-k}\rho(\mu)\,d\mu.$$

As above we have

$$R_3(\nu)\leqslant C\sum_{k=0}^{\infty}P(\nu+k)\int_{-\infty}^{-k}\rho(\mu)\,d\mu$$

$$\leqslant C\sum_{k=0}^{\infty}\frac{\nu+k}{(1+k)^N}\leqslant C\nu \quad \text{as } \nu\to\infty.$$

By the arguments above, we obtain

Theorem 1.7. *If we assume that* $P(\mu) = O(\mu)$ *holds as* μ *tends to* ∞, *then we find*

$$N(\nu) = \int_{-\nu}^{\nu} P(\mu)\, d\mu + O(\nu) \tag{1.7}$$

holds ν *tends to* ∞.

§ **2.** We say that a domain D in R^2 is a polygonal domain, when D is bounded and ∂D consistis of a finite number of segments. In this section we shall giue a representaion of $e(x, \xi, t)$ for small t in such a domain.

Let R be a rectangle containing D. Let $\{a_\alpha\}$ be a collection of mesh points in R and $\{\varphi_\alpha\}$ be a partition of unity in R such that supp $\varphi_\alpha \subset B(a_\alpha, \delta) = \{x \in R^2 \mid |x - a_\alpha| \leqslant \delta\}$ holds for each α where δ does not depend of α. Since the wave equation has the fiinte propagation speed, we can see that $\varphi_\alpha(x) e(x, \xi, t) = 0$ holds for $|\xi - a_\alpha| \geqslant T + \delta$ and $|t| \leqslant T$. Furthermore, when we choose beforehand sufficiently small δ and T, we may assume $B(a_\alpha, T+\delta)$ does not contain more than two corners of D.

Consequently for $x \in$ supp $\varphi_\alpha, \xi \in B(a_\alpha, T+\delta)$ and $t \leqslant T$, $\varphi_\alpha(x) e(x, \xi, t)$ can be identified with the solution of (1.1) in a sector or the free space. More precisely,let π/α be the angle of the corner contained in $B(a_\alpha, T+\delta)$ (we set $\alpha = 1/2$, if $B(a_\alpha, T+\delta)$ is contained in the interior of D) and $D_\alpha = \{(r\cos\theta, r\sin\theta) \in R^2 \mid 0 < r < \infty$ and $0 < \theta < \pi/\alpha\}$, then we obtain

Theorem 2.1. *Let* $e_\alpha(x, \xi, t)$ *be the solution of* (1.1) *in* $D = D_\alpha$, *then* $e(x, \xi, t)$ *is represeted as*

(2.1) $e(x, \xi, t) = \sum_\alpha \varphi_\alpha(x)\, e_\alpha(A_\alpha x + b_\alpha, A_\alpha \xi + b_\alpha, t)$ for $|t| \leqslant T$ *for sufficiently small* T, *where* A_α *is a rotation and* b_α *is a translation in* R^2.

In order to get the explicit form of $e_\alpha(x, \xi, t)$, we shall review the construction of the Green function in a sector following Friedlander [4], Ibuki [5] and [6].

Let $u(x, t)$ be the solution of the following problem

$$(2.2)\quad \begin{cases} \left(\dfrac{\partial^2}{\partial t^2} - \Delta\right) u(x,t) = 0 & x \in D_\alpha,\ t > 0 \\ u(x,0) = 0 & x \in D_\alpha \\ \dfrac{\partial}{\partial t} u(x,0) = \varphi(x) & x \in D_\alpha \\ u(x,t) = 0 & x \in \partial D_\alpha,\ t \geqslant 0, \end{cases}$$

then $u\ (x,\ t)$ is represented by the Green function $E_\alpha(x, \xi, t)$ as

$$u(x,t) = \int_{D_\alpha} E_\alpha\ (x, \xi, t)\ \varphi\ (x)\ dx.$$

We can see easily that $E_\alpha\ (x, \xi, t)$ is an odd fuction in t and that

$$(2.3)\qquad e_\alpha(x,\ \xi, t) = \left(\frac{\partial}{\partial t}\right) E\ (x, \xi, t).$$

We set $\varphi(x) = \varphi(r, \theta) = \sum_{=1}^{\infty} \varphi_n(r) \sin \alpha n\theta.$

We seek the solution $u(x,t)$ in the form

$$u(x,\ t) = u(r,\ \theta, t) = \sum_{n=1}^{\infty} u_n\ (r, t) \sin\ \alpha n\theta,$$

$$u_n\ (r, t) = \int_0^\infty A_n(\xi) \sin \xi t\ J_{\alpha n}(\xi r)\ d\xi,$$

where $J_\lambda\ (z)$ is the Bessel function of order λ. By virtue of Fourier-Bessel inversion formula. we find

$$A_n\ (\xi) = \int_0^\infty \varphi_n(r)\ J_{\alpha n}(\xi r)\ dr \quad \text{and then}$$

$$(2.4)\quad u_n(r, t) = \int_0^\infty \sin \xi t\, J_{\alpha n}\ (\xi r) \int_0^\infty J_{\alpha n}\ (\xi s)\ \varphi_n(s)\ s\ ds\ d\xi.$$

Furthermore

$$\begin{aligned} u_n(r,\ t) &= -\lim_{\varepsilon \to 0} \int_0^\infty Im\ e^{-(\varepsilon + it)\xi}\ J_{\alpha n}(\xi r) \int_0^\infty J_{\alpha n}\ (\xi s) \varphi_n(s)\ s\, ds\ d\xi \\ &= -\lim_{\varepsilon \to 0} \int_0^\infty \varphi_n\ (s)\ s \int_0^\infty Im\ e^{-(\varepsilon + it)\xi} J_{\alpha n}\ (\xi r) J_{\alpha n}\ (\xi s)\ d\xi\ ds. \end{aligned}$$

On the other hand, since we know the following formula (see Watson[8])

$$(2.5)\quad \int_0^\infty e^{-\zeta\xi}\ J_\lambda\ (\xi r)\ J_\lambda\ (\xi s)\ ds = \frac{1}{\pi\sqrt{rs}}\, Q_{-1/2}\left(\frac{\zeta^2 + r^2 + s^2}{2\,rs}\right)$$

where $Q_\lambda(z)$ is the Legendre function of the second kind, we have the following theorem after some careful limit process.

Theorem 2.2 (see Friedlander [4] and Ibuki [5]). *Let*

$$\text{(2.6} \quad K_n\ (r,s,t) = \begin{cases} 0 & , \quad 0 \leqslant t \leqslant |r-s| \\ \dfrac{1}{2\sqrt{rs}} P_{\alpha n-1/2}\left(\dfrac{-t^2+r^2+s^2}{2rs}\right), & |r-s| \leqslant t \leqslant r+s \\ \dfrac{\cos\alpha n}{\pi\sqrt{rs}} Q_{\alpha n-1/2}\left(\dfrac{t^2-r^2-s^2}{2rs}\right), & t < r+s, \end{cases}$$

then $u_n(r,t)$ *is represented as*

$$\text{(2.6)} \qquad u_n(r,t) = \int_0^\infty K_n(r,s,t)\varphi_n(s)s\,ds.$$

Since

$$u(r,\theta,t) = \sum_{n=1}^{\infty}\int_0^\infty K_n\ (r,s,t)\ \varphi_n\ (s)\ ds\ \sin\ \alpha n\theta$$

$$= \frac{2\alpha}{\pi}\sum_{n=1}^{\infty}\int_0^\infty\int_0^{\pi/\alpha} K_n\ (r,s,t)\ \sin\ \alpha n\theta\ \sin\ \alpha n\omega\ \varphi(s,\omega)\ sd\omega\,ds,$$

we find

$$\text{(2.7)} \quad E_\alpha(x,\xi,t) = E_\alpha(r,s,\theta,\omega,) = \frac{2\alpha}{\pi}\sum_{n=1}^{\infty} K_n(r,s,t)\ \sin\ \alpha n\theta\ \sin\ \alpha n\omega.$$

Here the infinite sum converges in appropriate distribution space. We shall show another representation of (2.7) following Friedlander [4] and Ibuki [6].

Proposition 2.3. *Let*

$$E\,(r,s,\theta,t) = \frac{1}{2\pi\sqrt{t^2-r^2-s^2+2rs\ \cos\theta}}$$

then for $|r-s| < t \leqslant r+s$ *we have*

$$\text{(2.8)} \quad E_\alpha\ (r,s,\theta,\omega,t)$$

$$= \sum_{|\theta-\omega+2\pi k/\alpha|<\lambda} E\,(r,s,\theta-\omega+2\pi k/\alpha,t) - \sum_{|\theta+\omega+2\pi k/\alpha|<\lambda} E\,(r,s,\theta+\omega+2\pi k/\alpha,t),$$

where $\quad \cos\lambda = \dfrac{-t^2+r^2+s^2}{2rs}.$

Outline of the proof. We know that $P_\lambda(\cos\theta)$ has the representation

$$P_\lambda(\cos\theta) = \frac{\sqrt{2}}{\pi}\int_0^\theta \frac{\cos(\lambda+1/2)}{\sqrt{\cos\varphi-\cos\theta}}d\varphi \quad (0<\theta<\pi)$$

(see for example Courant-Hilbert [3]). Then we find

$$E_\alpha(r, s, \theta, \omega, t) = \frac{2\alpha}{\pi}\sum_{n=1}^{\infty}\frac{1}{\pi\sqrt{2rs}}\int_0^\lambda \frac{\sin \alpha n\theta \ \sin \alpha n\omega \ \cos \alpha n\mu}{\sqrt{\cos\mu - \cos\lambda}}d\mu$$

where $\cos\lambda = \dfrac{-t^2+r^2+s^2}{2rs}$.

Since $\sin \alpha n\theta \sin \alpha n\omega \cos \alpha n\mu$

$= \frac{1}{4}[\cos \alpha n(\theta-\omega+\mu) + \cos \alpha n(\theta-\omega-\mu) - \cos \alpha n(\theta+\omega+\mu) - \cos \alpha n(\theta+\omega-\mu)]$

and $\frac{\pi}{\alpha}\mathring{\delta}(\theta) = \sum_{k=-\infty}^{\infty}\delta(\theta-2\pi k/\alpha) = \frac{1}{2} + \sum_{n=1}^{\infty}\cos \alpha n\theta$ hold, then we find

$E_\alpha(r, s, \theta, \omega, t)$

$$= \frac{1}{2\pi\sqrt{2rs}}[\mathring{\delta}(\theta-\omega+\mu) \underset{\mu}{*} \frac{H(|\mu|<\lambda)}{\sqrt{\cos\mu-\cos\lambda}} - \mathring{\delta}(\theta+\omega+\mu) \underset{\mu}{*} \frac{H(|\mu|<\lambda)}{\sqrt{\cos\mu-\cos\lambda}}]$$

which proves the proposition.

In a similar mannar we have

Proposition 2.4. *Let* $\tilde{E}(r,s,\theta,t,v)$

$$= \frac{\alpha}{4\pi^2}\left[\frac{1}{\cosh \alpha v - \cos \alpha(\pi+\theta)} + \frac{1}{\cosh \alpha v - \cos\alpha(\pi-\theta)}\right]\frac{\sinh \alpha v}{\sqrt{2rs\cosh v - t^2+r^2+s^2}}.$$

Then for $t > r+s$ *we have*

(2.9) $E_\alpha(r,s,\theta,\omega,t) = \int_u^\infty [\tilde{E}(r,s,\theta-\omega,t,v) - \tilde{E}(r,s,\theta+\omega,t,v)]dv$

where $\cosh u = \dfrac{t^2-r^2-s^2}{2rs}$.

Out line of the proof. As before we know that

$$Q_{\alpha n-1/2}(\cosh u) = \frac{1}{\sqrt{2}}\int_u^\infty \frac{e^{-\alpha v}}{\sqrt{\cosh u - \cosh v}}dv,$$

Since $\frac{1}{2}+\sum_{n=1}^{\infty} e^{-\alpha n v}\cos \alpha n\varphi = \dfrac{\sinh \alpha v}{2(\cosh \alpha v - \cos \alpha\varphi)}$ holds,

we can make use of the similar arguments as before and then have the proposition.

We can deform (2.9) to the form which is suitable for our applications.

Proposition 2.5. *Let $E(r,s,\theta,t)$ be the same as before and*

$D(r,s,\theta,t,v)$

$$= -\frac{\alpha}{4\pi^2}\left[\frac{\sin\alpha(\pi+\theta)}{\cosh\alpha v-\cos(\pi+\theta)}+\frac{\sin\alpha(\pi-\theta)}{\cosh\alpha v-\cos(\pi-\theta)}\right]\frac{1}{\sqrt{t^2-r^2-s^2-2rs\cosh v}}.$$

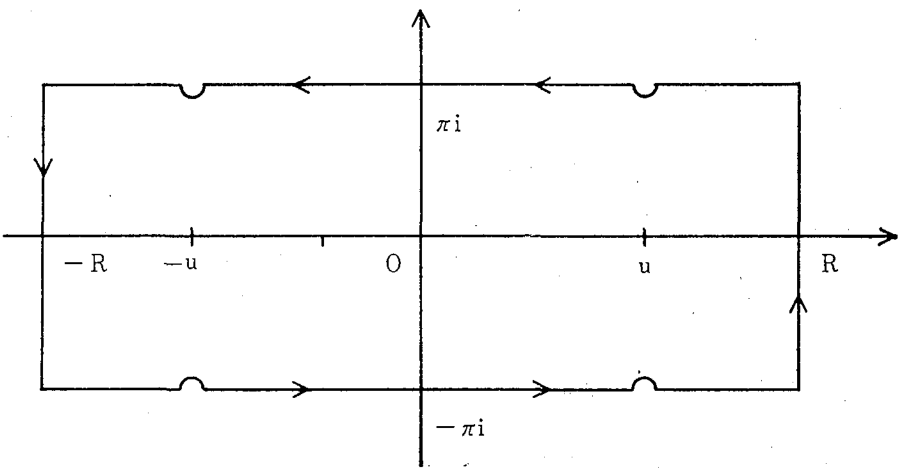
Then for $t < r+s$ we have

$$(2.10)\quad E_\alpha(r,s,\theta,\omega,t) = \sum_{|\theta-\omega+2\pi k/\alpha|<\pi} E(r,s,\theta-\omega+2k/\alpha,t)$$
$$-\sum_{|\theta+\omega+2\pi k/\alpha|<\pi} E(r,s,\theta+\omega+2\pi k/\alpha,t)$$
$$+\int_0^u D(r,s,\theta-\omega,t,v)-D(r,s,\theta+\omega,t,v)\; dv$$

Outline of the proof. We consider the integral

$$I_R = \frac{1}{2\pi i}\oint\frac{e^{\alpha v}}{e^{\alpha v}-e^{i\alpha}}\;\frac{1}{\sqrt{\cosh v+\cosh u}}\,dv$$

along the path



(Fig)

Since

$$I_R = \sum_{|\varphi+2\pi k/\alpha|<\pi} \operatorname*{Res}_{v=i(\varphi+\pi k/\alpha)} \frac{e^{-\alpha v}}{e^{\alpha v}-e^{i\alpha}} \frac{1}{\sqrt{\cosh v + \cosh u}},$$

we obtain (2.10) as $R\to\infty$ (see Friedlander [4] for the details).

Putting together Proposition 2.3, 2.4 and 2.5, we obtain

Theorem 2.6 (Friedlander [4] and Ibuki [6]). *Let*

$$E(r,s,\theta,t) = \frac{1}{2\pi\sqrt{t^2-r^2-s^2+2rs\cos\theta}} \quad \textit{and} \quad D(r,s,\theta,t,v)$$

$$= -\frac{\alpha}{4\pi^2}\left[\frac{\sin\alpha(\pi+\theta)}{\cosh\alpha v-\cos\alpha(\pi+\theta)}+\frac{\sin\alpha(\pi-\theta)}{\cosh\alpha v-\cos\alpha(\pi-\theta)}\right]\frac{1}{\sqrt{t^2-r^2-s^2-2rs\cosh v}}.$$

Then $E_\alpha(r,s,\theta,\omega,t)$ is represented as the following.

$$(2.11)\quad E_\alpha(r,s,\omega,t) = \sum_{|\theta-\omega+2\pi k/\alpha|<\lambda} H(t>|r-s|)E(r,s,\theta-\omega+2\pi k/\alpha,t)$$

$$-\sum_{|\theta+\omega+2\pi k/\alpha|<\lambda} H(t>|r-s|)E(r,s,\theta+\omega+2\pi k/\alpha,t)$$

$$+H(t<r+s)\int_0^u [D(r,s,\theta-\omega,t,v)-D(r,s,\theta+\omega,t,v)]dv$$

where $H(t)$ is the Heaviside function and $\cos\lambda = \max\left(\frac{-t^2+r^2+s^2}{2rs},-1\right)$,

$$\cosh u = \frac{t^2-r^2-s^2}{2rs}.$$

§ **3.** In the first place, we compute $\langle e_\alpha(x,x,t),\varphi(t)\rangle$. By (2.3), we find

$$(3.1)\quad \langle e_\alpha(x,\xi,t),\psi(t)\rangle$$

$$= -\int_0^\infty (\psi'(t)-\psi'(-t))E_\alpha(x,\xi,t)\,dt \ \text{ for } \psi\in C_0^\infty(R).$$

As $\psi'(t)-\psi'(-t)=O(t)$, the integral above makes sense in the ordinary way. Inserting (2.11) into (3.1) and exchanging the integral and the summation,

we find

$$\langle e_\alpha(x,\xi,t),\psi(t)\rangle =$$

$$-\sum_{|\theta-\omega+2\pi k/\alpha|<\pi}\frac{1}{2\pi}\int_{t^2\geqq r^2+s^2-2rs\cos(\theta-\omega+2\pi k/\alpha)}\frac{\psi'(t)-\psi'(-t)}{\sqrt{t^2-r^2-s^2+2rs\cos(\theta-\omega+2\pi k/\alpha)}}dt$$

$$+\sum_{|\theta+\omega+2\pi k/\alpha|<\pi}\frac{1}{2\pi}\int_{t^2\geqq r^2+s^2-2rs\cos(\theta+\omega+2\pi k/\alpha)}\frac{\psi'(t)-\psi'(-t)}{\sqrt{t^2-r^2-s^2+2rs\cos(\theta+\omega+2\pi k/\alpha)}}dt$$

$$+\frac{\alpha}{4\pi^2}\int_0^\infty\Big[\frac{\sin\alpha(\pi+\theta-\omega)}{\cosh\alpha v-\cos\alpha(\pi+\theta-\omega)}+\frac{\sin\alpha(\pi-\theta+\omega)}{\cosh\alpha v-\cos\alpha(\pi-\theta+\omega)}$$

$$-\frac{\sin\alpha(\pi+\theta+\omega)}{\cosh\alpha v-\cos\alpha(\pi+\theta+\omega)}-\frac{\sin\alpha(\pi-\theta-\omega)}{\cosh\alpha v-\cos\alpha(\pi-\theta-\omega)}\Big\}$$

$$\int_{t^2\geqq r^2+s^2+2rs\cosh v}\frac{\psi'(t)-\psi'(-t)}{\sqrt{t^2-r^2-s^2-2rs\cosh v}}\,dt\Big]dv.$$

Then setting $x=\xi$, we obtain

(3.2) $\langle e_\alpha(x,x,t),\ \psi(t)\rangle$

$$=-\frac{1}{2\pi}\int_0^\infty\frac{\psi'(t)-\psi'(-t)}{t}\,dt-\sum_{0<|\pi k/\alpha|<\pi/2}\frac{1}{2\pi}\int_{2r|\sin\pi k/\alpha\backslash}^{\infty}\frac{\psi'(t)-\psi'(-t)}{\sqrt{t^2-4r^2\sin^2\pi k/\alpha}}\,dt$$

$$+\sum_{0<|\theta+\pi/\alpha|<\pi/2}\frac{1}{2\pi}\int_{2r|\sin(\theta+\pi k/\alpha)|}^{\infty}\frac{\psi'(t)-\psi(-t)}{\sqrt{t^2-4r^2\sin^2(\theta+\pi k/\alpha)}}\,dt$$

$$-\frac{\alpha}{4\pi^2}\int_0^\infty\Big[\Big\{\frac{\sin\alpha(\pi+2\theta)}{\cosh\alpha v-\cos\alpha(\pi+2\rho)}+\frac{\sin\alpha(\pi-2\theta)}{\cosh\alpha v-\cos\alpha(\pi-2\theta)}-\frac{2\sin\alpha\pi}{\cosh\alpha v-\cos\alpha\pi}\Big\}$$

$$\int_{2r\cosh v/2}^{\infty}\frac{\psi'(t)-\psi'(-t)}{\sqrt{t^2-4r^2\cosh^2 v/2}}\,dt\Big]\,dv.$$

Furthermore inserting $\psi(t)=\frac{1}{2\pi}\,e^{it\mu}\rho(t)$ into (3.2,) we have

Theorem 3.1. *For $r>0$ and $0<\theta<\pi/\alpha$, we find*

(3.3) $\frac{1}{2\pi}\langle e_\alpha\,(x,x,t),e^{it\mu}\rho(t)\rangle$

$$=\frac{\mu}{2\pi^2}\int_0^\infty\frac{\rho(t)\sin t\mu}{t}\,dt+\frac{\mu}{2\pi^2}\sum_{0<\pi k/\alpha<\pi/2}\int_{2r|\sin\pi k/\alpha|}^{\infty}\frac{\rho(t)\sin t\mu}{\sqrt{t^2-4r^2\sin^2\pi k/\alpha}}\,dt$$

$$-\frac{\mu}{2\pi^2}\sum_{0<\theta+\pi k/\alpha<\pi/2}\int_{2r|\sin(\theta+\pi k/\alpha)|}^{\infty}\frac{\rho(t)\sin t\mu}{\sqrt{t^2-4r^2\sin^2(\theta+\pi k/\alpha)}}\,dt$$

$$+\frac{\alpha\mu}{4\alpha^3}\int_0^\infty\Big[\Big\{\frac{\sin\ \alpha(\pi+2\theta)}{\cosh\alpha v-\cos\alpha(\pi+2\theta)}+\frac{\sin\ \alpha(\pi-2\theta)}{\cosh\alpha v-\cos\alpha(\pi-2\theta)}-\frac{2\sin\ \alpha\pi}{\cosh\alpha v-\cos\alpha\pi}\Big\}$$

$$\int_{2r\cosh v/2}^\infty\frac{\rho(t)\ \sin\ t\mu}{\sqrt{\ t^2-4r^2\cosh v/2}}dt\Big]dv$$

$$-\frac{1}{2\pi^2}\sum_{|\pi k/\alpha|<\pi/2}\int_{2r|\sin\ \pi k/\alpha|}^\infty\frac{\rho'(t)\cos t\mu}{\sqrt{t^2-4r^2\sin^2\pi k/\alpha}}dt$$

$$+\frac{1}{2\pi^2}\sum_{0<\theta+\pi k/\alpha|<\kappa/2}\int_{2r|\sin(\theta+\pi k/\alpha)6}^\infty\frac{\rho'(t)\cos t\mu}{\sqrt{t^2-4r\sin\ (\theta+\pi k/\mu)}}dt$$

$$-\frac{\alpha}{4\pi^3}\int_0^\infty\Big[\Big\{\frac{\sin\ \alpha\ (\pi\ +\ 2\ \theta)}{\cosh\alpha v-\cos\alpha(\pi+2\theta)}+\frac{\sin\alpha(\pi-2\theta}{\cosh\alpha v-\cos\alpha(\pi-2\theta)}-\frac{2\sin\ \alpha\pi}{\cosh\alpha\ v-\cos\alpha\pi}\Big\}$$

$$\int_{2r\cosh v/2}^\infty\frac{\rho'(t)\ \cos\ t\mu}{\sqrt{t^2-\ 4r\cosh v/2}}dt\Big]dv.$$

Next we shall show several calculus lemmas which will be used later.

Lemma 3.2. *Let* $\rho\in C_0^\infty(R)$, *then we find*

$$\int_0^\infty\rho(t)\ e^{it\mu}\ dt=O(\frac{1}{\mu})\ \ for\ \ \mu>1. \tag{3.4}$$

Proof. By integration by parts, we get the estimate above.

Lemma 3.3. *Let* $\rho\in C_0^\infty(R)$ *and satisfy* $\rho(t)\geqslant 0$ *and* $\rho'(t)\geqslant 0$ *for* $t<0$, *then we find*

$$\int_a^\infty\frac{\rho(t)\ e^{it}}{\sqrt{t^2\ -\ a^2}}dt\ =\ O(\frac{1}{\sqrt{a\mu}})\ \ for\ \ 0\ <a<\frac{1}{2}\ \ and\ \ \mu>1. \tag{3.5}$$

Proof. We change the variables by setting $t=a+s\mu$, then we find

$$\int_a^\infty\frac{\rho\ (t)\ \ e^{it\mu}}{\sqrt{t^2\ -\ \alpha^2}}dt=\frac{e^{iat}}{\sqrt{\alpha\mu}}\int_0^\infty\frac{\rho(a+s/\mu)\ e^{st}}{\sqrt{\ s\ (2+s/\mu)}}ds$$

$$=O(\frac{1}{\sqrt{\alpha\mu}}\int_0^A\frac{1}{\sqrt{s}}ds)\ +O\Big(\frac{1}{\sqrt{a\mu}}\int_A^\infty\frac{\rho\ (a+s/\mu)\ \ e^{is}}{\sqrt{s(2+s/\mu)}}ds\Big).$$

Since $\rho(a+s/\mu)/s(2+s/\mu)$ is decreasing in s, we find by the second mean value theorem

$$\int_a^\infty\frac{\rho(t)\ e^{it}}{\sqrt{t^2-a^2}}dt=O\ (\frac{1}{\sqrt{a\mu}})+O\Big(\frac{\rho(a+A/\mu)}{\sqrt{a\mu}\sqrt{A(2+A/\mu)}}\sup_{\xi\geqslant A}|\int_A^\xi e^{is}ds|\Big)$$

$$=O\ (\frac{1}{\sqrt{a\mu}}).$$

Then we obtain the lemma.

Lemma 3.4.

$$(3.6)\quad \int_0^\infty \frac{\sin\beta}{(\cosh \alpha v-\cos\beta)\sqrt{\cosh v/2}}\,dv = O(1)$$

holds uniformly in β $(0\leqslant\beta\leqslant\pi/2)$.

Proof. We divide the integral into two parts as the following.

$$\int_0^\infty = \int_0^A + \int_A^\infty . \text{ since}$$

$$(3.7)\quad \left|\int_A^\infty\right| \leqslant \int_A^\infty \frac{1}{(\cosh\alpha v-1)\sqrt{\cosh v/2}}\,dv$$

$$\leqslant \frac{2\sqrt{2}}{1-e^{-\alpha A}}\int_A^\infty e^{-(\alpha+1/4)v}\,dv \leqslant C,$$

$$(3.8)\quad \left|\int_0^A\right| \leqslant \int_0^A \frac{|\sin\beta|}{(\cosh\alpha v-\cos\beta)}\,dv$$

$$\leqslant \int_0^A \frac{|\sin\beta|}{(\alpha^2v^2/2+1-\cos\beta)}\,dv$$

$$=\frac{2}{\alpha^2}\int_0^A \frac{|\sin\beta|}{(v^2+(4/\alpha^2)\sin^2\beta/2)}\,dv \leqslant \frac{1}{2\alpha}\,\frac{|\sin\beta|}{|\sin\beta/2|}\int_0^\infty\frac{du}{1+u^2}\leqslant C,$$

we obtain the lemma.

Now we carry out step by step the estimates of the each term in (3.3).

Proposition 3.5.

$$(3.9)\quad \frac{\mu}{2\pi^2}\int_0^\infty \frac{\rho(t)\sin t\mu}{t}\,dt = \frac{\mu}{4\pi} + O(1)$$

holds uniformly in $\mu>0$

Proof. By Lemma 3.2. we find

$$\frac{\mu}{2\pi^2}\int_0^\infty \frac{\rho(t)\sin t\mu}{t}\,dt = \frac{\mu}{2\pi^2}\int_0^\infty \frac{\sin t\mu}{t}\,dt + \frac{\mu}{2\pi^2}\int_0^\infty \frac{(1-\rho(t))\sin t\mu}{t}\,dt$$

$$=\frac{\mu}{2\pi^2}\int_0^\infty \frac{\sin t}{t}\,dt + O(1).$$

Proposition 3.6. *For* $r>0$ *and* $0<\theta<\pi/\alpha$, *we have*

$$(3.10)\ \frac{\mu}{2\pi^2}\Big[\int_{2r|\sin\pi k/\alpha|}^{\infty}\frac{\rho(t)\sin t\mu}{\sqrt{t^2-4r^2\sin^2\pi k/\alpha}}dt-\int_{2r|\sin(\theta+\pi k/\alpha)|}^{\infty}\frac{\rho(t)\sin t\mu}{\sqrt{t^2-4r^2\sin^2(\theta+\pi k/\alpha)}}dt\Big]$$
$$=0\Big(\sqrt{\frac{\mu}{r}}\frac{1}{\sqrt{|\sin(\theta+\pi k/\alpha)|}}\Big).$$

Proof. By Lemma 3.3, we can see

$$(3.11)\ \int_{2r|\sin\varphi|}^{\infty}\frac{\rho(t)\sin t\mu}{\sqrt{t^2-4r^2\sin^2\varphi}}dt = O\Big(\frac{1}{\sqrt{\mu r}\ |\sin\varphi|}\Big).$$

Then the statement follows from the estimate above.

Proposition 3.7

$$(3.12)\ \frac{\alpha\mu}{4\pi^3}\int_0^{\infty}\Big[\Big\{\frac{\sin\alpha(\pi+2\theta)}{\cosh\alpha v-\cos\alpha(\pi+2\theta)}+\frac{\sin\alpha(\pi-2\theta)}{\cosh\alpha v-\cos\alpha(\mu-2\theta)}-\frac{2\sin\alpha\mu}{\cosh\alpha v-\cos\alpha\mu}\Big\}$$
$$\int_{2r\cosh v/2}^{\infty}\frac{\rho(t)\sin t}{\sqrt{t^2-4r^2\cosh^2 v/2}}dt\Big]\,dv$$

$=O(\sqrt{\frac{\mu}{r}})$ *holds uniformly in* θ.

Proof. By Lemma 3.3, we find

$$\Big|\frac{\alpha\mu}{4\pi^3}\int_0^{\infty}\int_{2r\cosh v/2}^{\infty}dtdv\Big|$$
$$\leqslant C\sqrt{\frac{\mu}{r}}\int_0^{\infty}\Big|\frac{\sin\alpha(\pi+2\theta)}{\cosh\alpha v-\cos\alpha(\pi+2\theta)}+\frac{\sin\alpha(\pi-2\theta)}{\cosh\alpha v-\cos\alpha(\pi-2\theta)}-\frac{2\sin\alpha\pi}{\cosh\alpha v-\cos\alpha\pi}\Big|$$
$$\frac{1}{\sqrt{\cosh v/2}}dv$$

The we obtain (3.12) by Lemma 3.4.

Proposition 3.8.

$$(3.13)\ \frac{1}{2\pi^2}\int_{|2r\sin\pi k/\alpha|}^{\infty}\frac{\rho'(t)\cos t\mu}{\sqrt{t^2-4r^2\sin^2\pi k/\alpha}}dt-\int_{2r|\sin(\theta+\pi k/\alpha)|}^{\infty}\frac{\rho'(t)\cos t\mu}{\sqrt{t^2-4r^2\sin(\theta+\pi k/\alpha)}}dt$$

$= O(1)$ *and*

$$(3.14)\ \frac{\alpha}{4\pi^3}\int_0^{\infty}\frac{\sin\alpha(\pi+2\theta)}{\cosh\alpha v-\cos\alpha(\pi+2\theta)}+\frac{\sin\alpha(\pi-2\theta)}{\cosh\alpha v-\cos\alpha(\pi-2\theta)}-\frac{2\sin\alpha\pi}{\cosh\alpha v-\cos\alpha\pi}\Big\}$$
$$\int_{2r\cosh v/2}^{\infty}\frac{\rho'(t)\cos t\mu}{\sqrt{t^2-4r^2\cos^2 v/2}}dt\Big]\,dv$$

$= O(1)$ *holds uniformly in* μ, r *and* θ.

Proof. Since $\rho'(t)$ is an odd function, $|\rho'(t)| \leqslant Ct$ holds for $0 \leqslant t \leqslant T$. Furthermore $\int_a^T \frac{1}{\sqrt{t^2-a^2}}\,dt = \sqrt{T^2-a^2} = O(1)$ holds uniformly in small a. Then we can easily get the proposition.

With the aid of the preceeding propositions, we can show the main result of this section.

Theorem 3.9.

(3.15) $P(\mu) = \frac{|D|}{4\pi}\mu + O(\sqrt{\mu})$ *holds as* $\mu \to \infty$, *where* $|D|$ *denotes the area of the domain D.*

Proof. By definition, we have

$$P(\mu) = \frac{1}{2\pi}\int_D \langle e(x,x,t), e^{it\mu}\rho(t)\rangle\, dx$$

$$= \frac{1}{2\pi}\sum_\alpha \int_D \varphi_\alpha(x) \langle e_\alpha(Ax+b_\alpha,\ A_\alpha x+b_\alpha,\ t),\ e^{it\mu}\rho(t)\rangle\, dx.$$

Combining Proposition 3.5,3.6,3.7 and 3.8, we find

$$P(\mu) = \frac{1}{4\pi}\sum_\alpha \int_D \varphi_\alpha(x)dx + O(\sqrt{\mu}\sum_{\alpha,k}\int_0^\infty\int_0^{\pi/\alpha} \frac{\varphi_\alpha(r,\theta)\chi(0<|\theta+\pi k/\alpha|<\pi/2)}{\sqrt{|\sin(\theta+\pi k/\alpha)|}}\sqrt{r}\,d\mu\, dr)$$

$$= \frac{|D|}{4\pi}\mu + O(\sqrt{\mu}) \text{ as } \mu \to \infty.$$

In this way we obtain the theorem.

§ **4.** We proved that $P(\mu) \leqslant C\mu$ holds as $\mu \to \infty$ in the previous section.
We observe that the condition of Theorem 1.7 is satisfied. Then what we have to do is to get the asymptotic behavior of the first term of (1.7).

We must prepare a few more calculus lemmas.

Lemma 4.1. *Let* $\rho \in C_0^\infty(R)$, *then we have*

(4.1) $$\int_0^\nu \mu \int_{ar}^\infty \frac{\rho(t)\sin t\mu}{\sqrt{t^2 - a^2r^2}}\,dt\,d\mu = O\left(\frac{\nu}{ar}\right) .$$

If $\rho(t)=O(t^2)$ holds in addition, the right hand side of (4.1) *can be replaced with $O(\nu)$ which is uniform in a and r.*

Proof. Exchanging the the order of integrations in(4.1), we find

$$\int_0^\nu\int_{ar}^\infty dt\,d\mu=\int_{ar}^\infty\frac{\rho(t)}{\sqrt{t^2-a^2r^2}}\int_0^\nu\mu\sin t\mu\,d\mu\,dt$$

$$=-\int_{ar}^\infty\frac{\rho(t)}{t\sqrt{t^2-a^2r^2}}\,(\cos t\nu-\frac{1}{t\nu}\sin t\nu)\,dt.$$

Since $\cos t\nu-\frac{1}{t\nu}\sin t\nu=O(1)$ holds uniformly in t and ν, we find

$$(4.1)\quad \int_0^\nu\mu\int_{ar}^\infty\frac{\rho(t)\sin t\mu}{\sqrt{t^2-a^2r^2}}dt\,d\mu=O(\nu\int_{ar}^\infty\frac{\rho(t)}{t\sqrt{t^2-a^2r^2}}dt)=O\left(\frac{\nu}{ar}\right).$$

When we assume $\rho(t)=O(t^2)$, we find in a similar manner as above

$$(4.1)'\int_0^\nu\mu\int_{ar}^\infty\frac{\rho(t)\sin t\mu}{\sqrt{t^2-a^2r^2}}\,dt\,d\mu=O\,(\nu\int_{ar}^A\frac{t}{\sqrt{t^2-a^2r^2}}dt)=O(\nu).$$

Then we obtain the lemma.

Lemma 4.2. *Let $\varphi(r,\theta)\in C^\infty([0,\infty]\times[0,\pi/2])$ and $\equiv 0$ in a neighborhood of $\pi/2$. Let $\rho\in C_0^\infty(R)$ and satisfy $\rho(t)=1+O(t^2)$ for small t. Then we have for $\nu\geqslant 1$ and $0\leqslant r\leqslant r_0$*

$$(4.2)\quad \int_0^\nu\mu\int_0^{\pi/2}\int_{2r\sin\theta}^\infty\frac{\varphi(r,\theta)\rho(t)\sin t\mu}{\sqrt{t^2-4r^2\sin^2\theta}}\,dt\,d\theta\,d\mu=O\,(\frac{\nu}{r}).$$

Proof· We divide the inner integral as the following.

$$\int_{2r\sin\theta}^\infty dt=\int_{2r\sin\theta}^\infty\frac{\sin t\mu}{\sqrt{t^2-4r^2\sin^2\theta}}\,dt-\int_A^\infty\frac{\sin t\mu}{\sqrt{t^2-4r^2\sin^2\theta}}\,dt$$
$$+\int_{2r\sin\theta}^A\frac{(1-\rho(t))\sin t\mu}{\sqrt{t^2-4r^2\sin^2\theta}}\,dt$$
$$=\int_{2r\sin\theta}^\infty\frac{\sin t\mu}{\sqrt{t^2-4r^2\sin^2\theta}}\,dt+\int_{2r\sin\theta}^A\frac{(1-\rho(t))\sin t\mu}{\sqrt{t^2-4r^2\sin^2\theta}}\,dt+O\left(\frac{1}{\mu}\right).$$

Since an integral representation of the Hankel function of the first kind says

$$\int_1^\infty\frac{\sin tz}{\sqrt{t^2-1}}\,dt=Im\int_1^\infty\frac{e^{itz}}{\sqrt{t^2-1}}\,dt=Im\left(\frac{\pi}{2}iH_0^1(z)\right)=\frac{\pi}{2}J_0(z)$$

where $J_0(z)$ is the Bessel function of order 0, then we find that

$$\int_0^{\pi/2}\int_{2r\sin\theta}^{\infty}\frac{\varphi(r,\theta)\ \sin t\mu}{\sqrt{t^2-4r^2\sin^2\theta}}dt\ d\theta=\frac{\pi}{2}\int_0^{\pi/2}\varphi(r,\theta)J_0(2r\mu\sin\theta)\ d\theta$$

$$=\frac{\pi}{2}\int_0^1\frac{\varphi\ (r,\theta)}{\cos\theta}J_0\ (2r\mu s)\ ds$$

$$=\frac{1}{2r}\int_0^{2r\mu}\psi\ (r,\ \frac{z}{2r\ \mu}J_0\ (z)\ dz$$

Here we may take $\psi(r,s)=\frac{\pi}{2}\frac{\psi(r,\theta)}{\cos\theta}$ $(s=\sin\theta)$ to be be a smooth function of r and s, The Bessel function $J_0(z)$ is an analytic function and has the asymptotic behavior

$$J_0(z)=\sqrt{\frac{2}{\pi}}\,\frac{\cos(z-\pi/4)}{z^{1/2}}+O(z^{-3/2})\quad\text{as } z\to\infty. \tag{4.6}$$

Inserting (4.6) into (4.5) and thenby by integration by parts, we find

$$\int_0^{\pi/2}\int_{2r\sin\theta}^{\infty}\frac{\varphi(r,\ \theta)\ \sin t\mu}{\sqrt{t^2-4r^2\sin^2\theta}}dt\ d\mu \tag{4.7}$$

$$=\frac{1}{2r\mu}\int_0^A\psi J_0\ (z)\ dz\ +\ \frac{1}{2r\mu}\sqrt{\frac{2}{\pi}}\int_A^{2r\mu}\psi\frac{\cos\ (z-\pi/4)}{z^{1/2}}dz+O(\frac{1}{r\mu}\int_A^{2r\mu}z^{-3/2}\ dz)$$

$$=0(\frac{1}{r\mu}\int_A^{\infty}\psi\ (r,\frac{z}{2r\mu})\frac{\sin\ \ (z-\pi/4)}{z^{1/2}}dz)+O\Big(\frac{1}{r\mu}\int_A^{\infty}(\frac{z}{2r\mu})\ \frac{\partial\psi}{\partial s}(r,\ \frac{z}{2r\mu})\ \frac{\sin(z-\pi/4)}{z^{3/2}}$$

$$dz)\ +O(\frac{1}{r\mu})$$

$$=O(\frac{1}{r\mu}).$$

Multiplying (4.7) by μ and then integrating from 0 to ν, we obtain

$$\int_0^{\nu}\mu\int_0^{\pi/2}\int_{2r\sin\theta}^{\infty}\frac{\varphi(r,\ \theta)\ \sin t\mu}{\sqrt{t^2-4r^2\sin^2\theta}}dt\ d\theta\ d\mu\ =\ O(\frac{\nu}{r})\ . \tag{4.8}$$

Next we carry out the estimate of the second term of (4.3). Since $1-\rho(t)=O(t^2)$, multiplying the term by μ and integrating from 0 to ν, we find by
Lemma 4.1 that

$$\int_0^{\nu}\mu\int_{2r\sin\theta}^{A}\frac{(1-\rho(t))\sin t\mu}{\sqrt{t^2-4r^2\sin^2\theta}}dt\,d\mu=O(\nu). \tag{4.9}$$

Then multiplying (4.9) by φ and integrating from 0 to $\pi/2$ in θ, we obtain

$$(4.10)\quad \int_0^\nu \mu \int_0^{\pi/2}\int_{2r\sin\theta}^{A} \frac{(1-\rho(t))\sin t\mu}{\sqrt{t^2-4r^2\sin^2\theta}}\,dt d\theta d\mu = O(\nu).$$

In this way we obtain the lemma.

Now, with the aid of Lemma 4.1 aud 4.2, we carry out the estimate of

$$Q_\alpha(\nu) = \frac{1}{2\pi}\int_0^\nu\int_D \varphi_\alpha(x) < e_\alpha(A_\alpha x + b_\alpha, A_\alpha x + b_\alpha, t), t), e^{it\mu}\rho(t) > dx d\mu.$$

By Proposition 3.8,

$$\begin{aligned} Q_\alpha(\nu) = {} & \frac{\nu^2}{8\pi}\int_D \varphi_\alpha(x)\,dx + \frac{1}{4\pi^2}\sum_{0<\pi k/\alpha<\pi/2}\int_0^\nu \mu\int_0^\infty r\int_0^\pi\int_{2r\sin\pi k/\alpha}^\infty \\ & \qquad \frac{\varphi_\alpha(r,\theta)\ \rho(t)\ \sin t\mu}{\sqrt{t^2-4r^2\sin^2\pi k/\alpha}}\,dt\,d\theta\,dr\,d\mu \\ & - \frac{1}{4\pi^2}\sum_k\int_0^\nu \mu\int_0^\infty r\int_0^{\pi/\alpha} \chi\,(0<|\theta+\pi k/\alpha|<\pi k/2)\int_{2r|\sin(\theta+\pi k/\alpha)|}^\infty \\ & \qquad \frac{\varphi_\alpha(r,\theta)\rho(t)\ \sin t\mu}{\sqrt{t^2-4r^2\sin^2(\theta+\pi k/\alpha)}}\,dt\ d\theta\ dr\ d\mu \\ & + \frac{\alpha}{4\pi^3}\int_0^\nu \mu\int_0^\infty r\int_0^{\pi/\alpha}\int_0^\infty \Big[\Big\{\frac{\sin\alpha(\pi+2\theta)}{\cosh\alpha v-\cos\alpha(\pi+2\theta)} + \frac{\sin\alpha(\pi-2\theta)}{\cosh\alpha v-\cos\alpha(\pi-2\theta)} \\ & - \frac{2\sin\pi\alpha}{\cosh\alpha v - \cos\pi\alpha}\Big\}\int_{2r\cosh v/2}^\infty \frac{\varphi_\alpha(r,\theta)\rho(t)\sin t\mu}{\sqrt{t^2-4r^2\cosh^2 v/2}}\,dt\Big]\,dv\ d\theta\ dr\ d\mu \\ & + O(\nu). \end{aligned}$$

By Lemma 4.1 we find

$$(4.11)\int_0^\nu \mu\int_{2r|\sin\pi k/\alpha|}^\infty \frac{\rho(t)\ \sin t\mu}{\sqrt{t^2-4r^2\sin^2\pi k/\alpha}}\,dt\ d\mu = O\Big(\frac{\nu}{r\ |\sin\pi k/\alpha|}\Big)$$

$$(4.12)\int_0^\nu \mu\int_{2r\cosh v/2}^\infty \frac{\rho(t)\ \sin t\mu}{\sqrt{t^2-4r^2\ \cosh^2 v/2}}\,dt\ d\mu = O\Big(\frac{\nu}{r\ \cosh v/2}\Big).$$

Similarly by Lemma 4.2 we find, dividing the range of the integral if necessary

$$(4.13)\int_0^\nu \mu\int_0^{\pi/\alpha}\int_{2r\sin(\theta+\pi k/\alpha}^\infty \frac{\varphi^\alpha(r,\theta)\ \rho(t)\ \sin t\mu}{\sqrt{t^2-4r^2\sin^2(\theta+\pi\ k/\alpha)}}\,dt\ d\theta\ d\mu = O\Big(\frac{\nu}{r}\Big).$$

Then combining (4.11), (4.12) and (4.13), we obtain

Proposition 4.3.

$$\int_{-\nu}^\nu P(\mu)\ d\mu = \frac{|D|}{4\pi}\nu^2 + O(\nu)\ \textit{holds as}\ \nu\to\infty.$$

Proof. By (4.11), (4.12) and (4.13), we find

$$Q_\nu(\nu) = \frac{\nu^2}{8\pi}\int_D \varphi(x)\,dx + O(\nu).$$

Since $P(\nu)$ is an even function, we have

$$\int_{-\nu}^{\nu} P(\mu)\,d\mu = 2\sum_\alpha Q_\alpha(\nu) = \frac{|D|}{4\pi}\nu^2 + O(\nu).$$

Then we obtain the proposition.

By Proposition 4.3 together with Theorem 1.7, we obtain

Theorem 4.4. *Let $N(\nu)$ be the number of the eigenvalues satisfying $\sqrt{\nu_j} \leqslant \nu$. Then $N(\nu)$ has the asymptotic behavior of the form*

$$N(\nu) = \frac{|D|}{4\pi}\nu^2 + O(\nu) \quad as \ \nu \to \infty.$$

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