ON WOLD-WHITTLE'S MODEL FOR WEALTH DISTRIBUTION

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§ 0.

It is empirically known that the distribution of wealth W for higher estate levels obeys the Pareto law, that is, the density of distribution is propotional to a power of wealth $W^{-\alpha-1}(\alpha>1)$. In [1], H. O. A. Wold and P. Whittle proposed a model for the distribution of wealth explaining the Pareto law. Let x denote the size of an estate, $N(x, t)$ the number of persons at time t whose estates are less than x and $f(x, t)$ the density of $N(x, t)$. They supposed that an estate of size x grows to size $x(1+\beta\Delta t)+o(\Delta t)$ during a time interval Δt, that any person has possibility $\gamma\Delta t+o(\Delta t)$ of dying in the time interval Δt and on his death his estate is divided equally among n inheritors. Such a model would be applicable to persons with relatively large estates. Assuming also that $f(x, t)$ is differentiable in x and t, they showed that $f(x, t)$ satisfies the partial differential equation (0. 1) for large x. A derivation of the equation is found in [1] (see also [3]).

In this paper we assume that their assumptions are valid for every estate levels, which may be far from the reality, and study the density $f(x, t)$ as a solution of the following initial-value problem for the partial differential equation

$$\text{(0. 1)}\quad \frac{\partial f}{\partial t}(x, t)=-\beta x\frac{\partial f}{\partial x}(x, t)-(\beta+\gamma)f(x, t)+\gamma n^2 f(nx, t)$$

$$\text{for } x>0 \text{ and } t>0,$$

$$\text{(0. 2)}\quad f(x, 0)=f_0(x) \quad \text{for } x>0.$$

In §1, we give an exact solution of (0. 1) and (0. 2). A heuristic argument for obtaining the solution will be shown in Appendix A. In §2, we ensure that the solution of the intial-value problem is unique and then coincides with the solution of the previous section. In §3, we study the asymptotic behavior of the solution as time t tends to infinity.

A proof of an auxiliary lemma, which is crucial in determinig the asymptotic behavior, is carried out in Appendix B. Our results on the asymptotic behavior seem to have little value from the economical viewpoint. In order to squeeze any meaningful fact out of our mathematical results, we must require certain normalizations for $f(x, t)$. Such discussions are to be carried out in the forth coming paper [3].

In conclusion, I would like to thank Professor S. Watanabe who informed me of this interesting model and gave me incessant encouragements.

§ 1.

Let us denote by $C^n(D)$ the class of continuous functions which have derivatives up to order n in a domain D. We can write down an exact solution of the initial-value problem for a differentiable initial data f_0 as the following.

Theorem 1. 1. *Let* $f_0 \in C^1((0, \infty)) \cap C^0([0, \infty))$ *satisfy* $|f_0(x)|$, $|f_0'(x)| \leqslant L$ *for all* $x \in [0, \infty)$ *with some constant* L. *We define* $f(x, t)$ *as an infinite sum*

$$(1.\ 1)\quad f(x, t) = e^{-(\beta+\gamma)t} \sum_{k=0}^{\infty} \frac{(\gamma n^2 t)^k}{k!} f_0(n^k x e^{-\beta t}),$$

then f *belongs to* $C^1((0, \infty) \times (0, \infty)) \cap C^0([0, \infty) \times [0, \infty))$ *and is a solution of the initial-value problem. Moreover* $\sup_{x>0} |f(x, t)| \leqslant L e^{-(\beta+\gamma-\gamma n^2)t}$ *holds for all* $t > 0$.

Proof. Since $|f_0(x)| \leqslant L$ for all $x \in [0, \infty)$, the infinite sum (1. 1) converges uniformly in x and t on any bounded set in $[0, \infty) \times [0, \infty)$, and $\sup_{x>0} |f(x, t)| \leqslant L e^{-(\beta+\gamma-\gamma n^2)t}$ holds for all t. We find that f is continuous on $[0, \infty) \times [0, \infty)$ and satisfies the initial condition (0. 2). Similarly, since $|f_0'(x)| \leqslant L$ for all x, we are allowed to differentiate $f(x, t)$ term by term. Then we can see easily that $f(x, t)$ satisfies the equation (0. 1).

We can well expect that the next proposition holds under our assumptions. However, we shall not use it in this paper.

Proposition 1. 2. *If* $f_0 \in C^0([0, \infty))$ *has the following properties* i) $f_0(x) \geqslant 0$, ii) $\int_0^\infty f_0(x)\,dx < \infty$ (*the total population is finite*) *and* iii) $\int_0^\infty x f_0(x)\,dx < \infty$ (*the total wealth is finite*), *then* $f(x, t)$ *satisfies*

i) $f(x, t) \geqslant 0$ *for all* t,

ii) $\int_0^\infty f(x, t)\,dx = e^{\gamma(n-1)t}\int_0^\infty f_0(x)\,dx$,

iii) $\int_0^\infty xf(x, t)\,dx = e^{\beta t}\int_0^\infty xf_0(x)\,dx$.

Proof. i) is evident. Since

$$\sum_{k=0}^\infty \frac{(\gamma n^2 t)^k}{k!}\int_0^\infty f_0(n^k x e^{-\beta t})\,dx = \sum_{k=0}^\infty \frac{(\gamma n t)^k}{k!} e^{\beta t}\int_0^\infty f_0(x)\,dx$$

$$= e^{(\beta+n\gamma)t}\int_0^\infty f_0(x)\,dx,$$

ii) follows from Lebesgue's convergence theorem. We can prove iii) just in the same manner.

§ 2.

In this section, we shall prove the following theorem.

Theorem 2. 1. *The bounded solution of the initial-value problem is unique.*

Proof. We shall show that a solution $f(x, t) \in C^1((0, \infty) \times [0, \infty)) \cap C^0([0, \infty))$ of the problem satisfying $\sup_{0 \leqslant t \leqslant T} (\sup_{x \geqslant 0} |f(x, t)|) \leqslant L(T)$ for all $T \geqslant 0$ and $f(x, 0) \equiv 0$ must be identically zero.

When we set $g(x, t) = e^{(\beta+\gamma)t} f(x, t)$, $g(x, t)$ satisfies the following.

$$\text{(2. 1)} \quad \begin{cases} \dfrac{\partial g}{\partial t}(x, t) = -\beta x \dfrac{\partial g}{\partial x}(x, t) + \gamma n^2 g(nx, t) & \text{for } x > 0 \text{ and } t > 0 \\ g(x, 0) = 0 & \text{for all } x \geqslant 0. \end{cases}$$

Then we have only to show that $g(x, t)$ is identically zero. We proceed in the same manner as in proving Harr's inequallity for hyperbolic equations (see Chap. II, Appendix 1, §3 of [2]).

Since $x = \alpha e^{\beta t}$ is the characteristic curve starting from $x = \alpha$, we find by differetiating $g(x, t)$ along the curve that

$$\frac{d}{dt} g(\alpha e^{\beta t}, t) = \frac{\partial g}{\partial t} + \beta x \frac{\partial g}{\partial x}$$

$$= \gamma n^2 g(n\alpha e^{\beta t}, t).$$

Then we can see that g satisfies the following integral equation along the characteristic curve.

$$\text{(2. 2)} \quad g(\alpha e^{\beta t}, t) = \gamma n^2 \int_0^t g(n\alpha e^{\beta t}, s)\,ds.$$

When we set $G(t)=\sup_{x\geq 0}|g(x,t)|$, we find by running α from 0 to ∞ that $G(t)$ is finite for all t and satisfies the following integral inequality.

(2. 3) $\quad G(t)\leqslant \gamma n^2\int_0^t G(s)\,ds.$

By inserting (2. 3) into itself over and over again, we have

$$\begin{aligned} G(t) &\leqslant (\gamma n^2)^N\int_0^t\int_0^{t_{N-1}}\cdots\int_1^{t_1} G(s)\,ds dt_1 dt_2\cdots dt_{N-1} \\ &\leqslant \frac{(\gamma n^2 t)^N}{N!}\sup_{0\leqslant s\leqslant t} G(s)\leqslant \frac{(\gamma n^2 t)^N}{N!}L(t). \end{aligned}$$

Since N is arbitrary, we observe that $G(t)$ must be identically zero. Then we obtain the theorem.

Once we have established the uniqueness of the solution, we can take $f(x,t)$, defined as (1. 1), to be the unique solution for the initial-value problem.

§ 3.

In this section, we discuss the asymptotic behavior of the solution. Let $f(x)$ and $g(x)$ be any positive functions. We denote $f(x)\approx g(x)$ as $x\to a$, when there exist positive constants A and B such that $Af(x)\leqslant g(x)\leqslant Bf(x)$ holds for all x in a neighborhood of a. Next lemma is crucial in studying the asymptotic behavior of the solution, but the proof is put off until Appendix B. From now on, we shall denote the exponential of x interchangeably by e^x or $\exp(x)$.

Lemma 3. 1. *Let* $f_A(x)=\sum_{k=0}^{[Ax]}\frac{x^k}{k!}$, *where* $[a]$ *denotes the largest integer not exceeding* a. *Then we obtain*

i) $f_A(x)\approx e^x$ *as* $x\to\infty$, *if* $A\geqslant 1$,
ii) $f_A(x)\approx x^{-1/2}e^{(A-A\log A)x}$ *as* $x\to\infty$, *if* $0<A<1$,
iii) $e^x-f_A(x)\approx e^x$ *as* $x\to\infty$, *if* $0<A\leqslant 1$,
iv) $e^x-f_A(x)\approx x^{-1/2}e^{(A-A\log A)x}$ *as* $x\to\infty$, *if* $A<1$.

From now on, we shall confine ourselves with the initial data which have the following properties.

(3. 1) i) $f_0(x)>0$ for $0<x<\infty$,
ii) $f_0(x)\approx x^{-q}(q>0)$ as $x\to\infty$,
iii) $f_0(x)\approx x^p(p\geqslant 0)$ as $x\to 0$.

Theorem 3. 2. *If the initial data* $f_0(x)\in C^0([0,\infty))$ *satisfies* (3. 1), *then for any bounded interval* $[a, b]$ *in* $[0,\infty)$ *the solution* $f(x, t)$ *has the following asymptotic behavior which is valid uniformly in* $x\in[a, b]$.

i) *If* $\gamma n^{2+p}\log n\leqslant\beta$ *holds, then*

$$f(x, t)\approx\exp(-t(\beta+p\beta+\gamma-\gamma n^{2+p}))\quad as\quad t\to\infty,$$

ii) *If* $\gamma n^{2-q}\log n<\beta<\gamma n^{2+p}\log n$ *holds, then*

$$f(x, t)\approx t^{-1/2}\exp\left(-t\left(\beta+\gamma-\frac{\beta}{\log n}\left(1-\log\frac{\beta}{\gamma n^2\log n}\right)\right)\right)as\quad t\to\infty,$$

iii) *If* $\beta\leqslant\gamma n^{2-q}\log n$ *holds, then*

$$f(x, t)\approx\exp(-t(\beta-q\beta+\gamma-\gamma n^{2-q}))\quad as\quad t\to\infty.$$

Proof. We recall that the solution is represented as

$$f(x, t)=e^{-(\beta+\gamma)t}\sum_{k=0}^{\infty}\frac{(\gamma n^2t)^k}{k!}f_0(n^kxe^{-\beta t}).$$

We may assume $n>1$, for the asymptotic behavior is evident in the case $n\leqslant 1$. For sufficiently large N, we divide the summation into three parts as the following.

$$\sum_{k=0}^{\infty}=\sum_{k=0}^{[\beta t/\log n]-N}+\sum_{k=[\beta t/\log n]-N+1}^{[\beta t/\log n]+N}+\sum_{k=[\beta t/\log n]+N}^{\infty}$$
$$=F_1(x, t)+F_2(x, t)+F_3(x, t)\quad\text{for abbreviation.}$$

(I) Behavior of F_1.

Since $k\leqslant[\beta t/\log n]-N$ in this case, we have

$$0<n^kxe^{-\beta t}\leqslant be^{-N\log n}\quad\text{for}\quad x\in[a, b].$$

We may assume that N is so large that $f_0(n^kxe^{-\beta t})\approx n^{pk}e^{-p\beta t}$ holds for all k and t. Then we find

$$F_1(x, t)\approx e^{-p\beta t}\sum_{k=0}^{[\beta t/\log n]-N}\frac{(\gamma n^{2+p}t)^k}{k!}\quad\text{uniformly in } x\in[a, b]\ \text{ as } t\to\infty.$$

Since $\sum_{k=0}^{[\beta t/\log n]-N}=\sum_{k=0}^{[\beta t/\log n]}-\sum_{k=[\beta t/\log n]-N+1}^{[\beta t/\log n]}$
$=F_{11}+F_{12}$ again for abbreviation, by applying Lemma 3. 1 to F_{11} and Stirling's formula to F_{12}, we have

$$F_{11}(t)\approx\exp(t(\gamma n^{2+p}-\beta p))\quad\text{and}$$

$$F_{12}(t)\approx t^{-1/2}\exp\left(t\left(\frac{\beta}{\log n}\left(1-\log\frac{\beta}{\gamma n^{2+p}\log n}\right)-\beta p\right)\right)\quad\text{as}\quad t\to\infty$$

for $\gamma n^{2+p}\log n\leqslant\beta$. However we can see

$$\left(\frac{\beta t}{\log n}\left(1-\log\frac{\beta}{\gamma n^{2+p}\log n}\right)\right)\leqslant\gamma n^{2+p}t,\quad\text{we obtain}$$

(3. 2) $F_1(x, t)\approx\exp(t(\gamma n^{2+p}-p\beta))$ uniformly in $x\in[a, b]$ as $t\to\infty$, if $\gamma n^{2+p}\log n>\beta$.

In a similar manner, we obtain

$$(3.\ 3)\quad F_1(x,\ t)\approx t^{-1/2}\exp\left(\frac{\beta t}{\log n}\left(1-\log\frac{\beta}{\gamma n^2\log n}\right)\right)\quad\text{as}\quad t\to\infty,\quad \text{if}\quad \gamma n^{2+\rho}\log n>\beta.$$

(II) Behavior of F_2.

Since $[\beta t/\log n]-N+1\leqslant k\leqslant[\beta t/\log n]+N-1$, we find $0<ae^{-N\log n}\leqslant n^k xe^{-\beta t}\leqslant be^{N\log n}$. Here we may assume $f_0(n^k xe^{-\beta t})\approx 1$. Hence

$$F_2(x,\ t)\approx\sum_{k=[\beta t/\log n]-N+1}^{[\beta t/\log n]+N-1}\frac{(\gamma n^2 t)^k}{k!}\approx\frac{(\gamma n^2 t)^{[\beta t/\log n]}}{[\beta t/\log n]!},$$

we obtain by Stirling's formula

$$(3.\ 4)\quad F_2(x,\ t)\approx t^{-1/2}\exp\left(\frac{\beta t}{\log n}\left(1-\log\frac{\beta}{\gamma n^2\log n}\right)\right)\quad\text{as}\quad t\to\infty.$$

(III) Behavior of F_3.

Since $k\geqslant[\beta t/\log n]+N$, we find $n^k xe^{-\beta t}\geqslant ae^{(N-1)\log n}$ for all $x\in[a,\ b]$, we may assume $f_0(n^k xe^{-\beta t})\approx n^{-qk}e^{-\beta t}$ for all t and k. By employing the same arguments as in (I), we obtain

$$(3.\ 5)\quad F_3(x,\ t)\approx t^{-1/2}\exp\left(\frac{\beta t}{\log n}\left(1-\log\frac{\beta}{\gamma n^2\log n}\right)\right)\quad as\quad t\to\infty,\quad \text{if}\quad \beta>\gamma n^{2-q}\log n.$$

$$(3.\ 6)\quad F_3(x,\ t)\approx\exp\left(t(\gamma n^{2-q}+q\beta)\right)\quad\text{as}\quad t\to\infty,\quad\text{if}\quad\beta\leqslant\gamma n^{2-q}\log n.$$

Now we can determine the asymptotic behavior of the solution. Since $\gamma n^{2+\rho}\log n>\gamma n^{2-q}\log n$, we observe that only the three cases in the statement of the theorem take place.

i) *If* $\gamma n^{2+\rho}\log n\leqslant\beta$ holds, we find by (3. 2), (3. 4) and (3. 5)

$$f(x,\ t)\approx\exp\left(-t(\beta+p\beta+\gamma-\gamma n^{2+\rho})\right)+t^{-1/2}\exp\left(-t\left(\beta+\gamma-\frac{\beta}{\log n}\left(1-\log\frac{\beta}{\gamma n^2\log n}\right)\right)\right)\approx\exp\left(-t(\beta+p\beta+\gamma-\gamma n^{2+\rho})\right)\quad\text{as}\quad t\to\infty\quad\text{for all}\quad x\in[a,\ b].$$

ii) If $\gamma n^{2-q}\log n<\beta<\gamma n^{2+\rho}\log n$ holds, we find by (3. 3), (3. 4) and (3. 5)

$$f(x,\ t)\approx t^{-1/2}\exp\left(-t\left(\beta+\gamma-\frac{\beta}{\log n}\left(1-\log\frac{\beta}{\gamma n^2\log n}\right)\right)\right)$$

as $t\to\infty$ for all $x\in[a,\ b]$.

iii) If $\beta\leqslant\gamma n^{2-q}\log n$ holds, we find by (3. 3), (3. 4) and (3. 6)

$$f(x,\ t)\approx\exp\left(-t(\beta-q\beta+\gamma-\gamma n^{2-q})\right)$$

$$+t^{-1/2}\exp\left(-t\left(\beta+\gamma-\frac{\beta}{\log n}\left(1-\log\frac{\beta}{\gamma n^2\log n}\right)\right)\right)$$

$$\approx \exp\left(-t(\beta-q\beta+\gamma-\gamma n^{2-q})\right) \text{ as } t\to\infty \text{ for all } x\in[a, b].$$

In this way, we have proved the theorem.

Appendix A.

Here we show how we can find out a solution of (0. 2).

When we set $f(x, t)=e^{-(\beta+\gamma)t}g(\log x, t)$, is a solution of the following initial-value problem for a difference-differential equation.

$$\text{(A. 1)}\begin{cases}\dfrac{\partial g}{\partial t}(y, t)=-\beta\dfrac{\partial g}{\partial y}(y, t)+\gamma n^2 g(y+a, t) \text{ for } (y, t)\in(-\infty,\infty)\times(0,\infty)\\ g(y, 0)=g_0(y) \text{ for } y\in(-\infty, \infty), \text{ where}\end{cases}$$

(A. 2) $y=\log x$, $g_0(y)=f_0(e^y)$ and $a=\log n$.

Taking the Fourier transform of the both sides, we have the following differential equation with a parameter.

$$\text{(A. 3)}\begin{cases}\dfrac{d}{dt}g^\wedge(\xi, t)=(-i\beta\xi+\gamma n^2 e^{ia\xi})g^\wedge(\xi, t) \text{ for } t>0\\ g^\wedge(\xi, 0)=g_0^\wedge(\xi).\end{cases}$$

Here $g^\wedge$ denotes the Fourier transform of g.

The solution of (A. 3) is unique and has the form

$$\text{(A. 4)}\quad g^\wedge(\xi, t)=\exp\left(t(-i\beta\xi+\gamma n^2 e^{ia\xi})\right)g_0^\wedge(\xi)$$

$$=\sum_{k=0}^{\infty}\frac{(\gamma n^2 t)^k}{k!}e^{i\xi(ka-\beta t)}g_0^\wedge(\xi).$$

Then by taking the inverse Fourier transform of the both sides, we have

$$\text{(A. 5)}\quad g(y, t)=\sum_{k=0}^{\infty}\frac{(\gamma n^2 t)^k}{k!}g_0(y+ak-\beta t)$$

By inserting (A. 2) into (A. 5), we obtain the solution of the original initial-value problem.

Appendix B.

Here we carry out the proof of Lemma 3. 1.

i) Since $A\geqslant 1$ in this case, $f_1(x)\leqslant f_A(x)\leqslant e^x$ holds for $x>0$. we have only to show that $ce^x\leqslant f_1(x)$ holds for sufficiently large x with a positive constant c.

$$f_1(x)=\frac{x^{[x]}}{[x]!}+\frac{x^{[x]-1}}{([x]-1)!}+\cdots\cdots+\frac{x}{1!}+1$$
$$=\frac{x^{[x]}}{[x]!}\left\{1+\frac{[x]}{x}+\frac{[x]([x]-1)}{x^2}+\cdots+\frac{[x]([x]-1)\cdots 2\cdot 1}{x^{[x]}}\right\}$$
$$\geqslant\frac{x^{[x]}}{[x]!}\left\{1+\left(1-\frac{1}{x}\right)+\left(1-\frac{1}{x}\right)\left(1-\frac{2}{x}\right)+\cdots\right.$$
$$\left.+\left(1-\frac{1}{x}\right)\left(1-\frac{2}{x}\right)\cdots\left(1-\frac{[x]-1}{x}\right)\right\}.$$

We need only the first $[\sqrt{x}]$ terms of the summation. Then we have

(B. 1) $$f_1(x)\geqslant\frac{x^{[x]}}{[x]!}\left\{1+\left(1-\frac{1}{x}\right)+\left(1-\frac{1}{x}\right)\left(1-\frac{2}{x}\right)+\cdots\right.$$
$$\left.+\left(1-\frac{1}{x}\right)\left(1-\frac{2}{x}\right)\cdots\left(1-\frac{[\sqrt{x}]-1}{x}\right)\right\}$$

By expanding $\log(1+x)$ in Taylor series at $x=0$, we find $\log(1+x)=x-h(x)x^2$ for $|x|\leqslant 1/2$, where $0<h(x)\leqslant 1$. Then, since $1+x=e^{x-h(x)x^2}\geqslant e^{x-x^2}$ holds for $|x|<1/2$, we have

$$\left(1-\frac{1}{x}\right)\left(1-\frac{2}{x}\right)\cdots\left(1-\frac{n}{x}\right)$$
$$\geqslant\exp\left(\sum_{k=1}^{n}\frac{k}{x}-\sum_{k=1}^{n}\frac{k^2}{x^2}\right)$$
$$=\exp\left(-\frac{n(n+1)}{2x}-\frac{n(n+1)(2n+1)}{6x^2}\right)\quad\text{for } |x|<1/2.$$

As n runs from 0 to $[\sqrt{x}]-1$, we observe

(B. 2) $$\left(1-\frac{1}{x}\right)\left(1-\frac{2}{x}\right)\cdots\left(1-\frac{n}{x}\right)\geqslant\frac{1}{2}\exp\left(-\frac{n^2}{2x}\right)$$ holds uniformly in n for sufficiently large x. By inserting (B. 2) into (B. 1), we have

$$f_1(x)\geqslant\frac{x^{[x]}}{2[x]!}\left\{1+\exp\left(-\frac{1^2}{2x}\right)+\exp\left(-\frac{2^2}{2x}\right)+\cdots+\exp\left(-\frac{([\sqrt{x}]-1)^2}{2x}\right)\right\}$$
$$=\frac{x^{[x]+1/2}}{2[x]!}\cdot\frac{1}{\sqrt{x}}\sum_{n=0}^{[\sqrt{x}]-1}\exp\left(-\frac{1}{2}\left(\frac{n}{\sqrt{x}}\right)^2\right).$$

Since the sum $\frac{1}{\sqrt{x}}\sum_{n=0}^{[\sqrt{x}]-1}\exp\left(-\frac{1}{2}\left(\frac{n}{\sqrt{x}}\right)^2\right)$ converges to the integral $\int_0^1\exp\left(-\frac{t^2}{2}\right)dt$ as $x\to\infty$, we can see that there exists a positive constant c such that

$f_1(x) \geqslant c\frac{x^{[x]+1/2}}{[x]!} \geqslant c\frac{[x]^{[x]+1/2}}{[x]!}$ holds for large x.

By applying Stirling's formula, we obtain

$f_1(x) \geqslant c'e^{[x]} \geqslant c'e^{-1}e^{x}$ for sufficiently large x with another constant c'.

ii) As before

$$f_A(x) = \frac{x^{[Ax]}}{[Ax]!} + \frac{x^{[Ax]-1}}{([Ax]-1)!} + \ldots + \frac{x}{1!} + 1$$

$$= \frac{x^{[Ax]}}{[Ax]!}\left\{1 + \frac{[Ax]}{x} + \frac{[Ax]([Ax]-1)}{x^2} + \ldots + \frac{[Ax]([Ax]-1)\ldots 1}{x^{[Ax]}}\right\}$$

$$\leqslant \frac{x^{[Ax]}}{[Ax]!}(1 + A + A^2 + \ldots + A^{[Ax]}) \leqslant \frac{x^{[Ax]}}{(1-A)[Ax]!},$$

because $0 < A < 1$. On the other hand, $f_A(x) \geqslant \frac{x^{[Ax]}}{[Ax]!}$ holds evidenly. Then by applying Stirling's formula, we obtain the result.

Since the proofs of iii) and iv) proceed in the same way as above, we may omit the proof. In this way we have proved the lemma.

References.

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