THE BROUWER FIXED POINT THEOREM AND RELATED THEOREMS*

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In this paper it will be shown that the Brouwer fixed point theorem has many variant forms. The Brouwer fixed point theorem is equivalent to other related theorems; Sperner's lemma, the Knaster-Kuratowski-Mazurkiewicz theorem (K. K. M. theorem), No Retraction theorem, Variational Inequalities, Nonlinear Complementarity Problem, and the existence theorem of Walrasian equilibrium. To investigate such a relation will serve to study the internal structure of many problems in economics and Operations Research and to solve these problems.

1.

We use the following notation.

$N=\{0, 1, \ldots, n\}$: the set of integers

R^n : the n-dimensional Euclidean space

R^n_+ : the nonnegative orthant in R^n

$s=\{a^0, a^1, \ldots, a^n\}$: a set of points in R^n

$[s]=[a^0, a^1, \ldots a^n]$: the closed simplex spanned by s

$[s]_i=[a^0, \ldots, a^{i-1}, a^{i+1}, \ldots, a^n]$: the face opposite to the vertex a^i

$(s)=(a^0, a^1, \ldots, a^n)$: the open simplex spanned by s

Let's consider the first group of theorems.

(I) [Sperner's lemma] (Todd [10])

Let K be a triangulation of $[s]$ with each vertex of K labelled with an integer in N such that no vertex in $[s]_i$ is labelled i. (Such a labelling is called admissible or proper.) Then there is a simplex in

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K whose vertices carry all the labels in N. (Such a simplex is called a completely labelled simplex.)

(II) [K. K. M. theorem] (Kuratowski [6])

If A_i, $(i \in N)$, are closed sets such that each face $[t]=[a^{i_0}, a^{i_1}, \ldots, a^{i_k}]$, $(k \leqq n)$, of the simplex $[s]$ satisfies

$$[t] \subset A_{i_0} \cup \ldots \cup A_{i_k},$$

then $\bigcap_{i \in N} A_i \neq \phi$.

(II′) If A_i, $(i \in N)$, are closed sets such that

(i) $[s]=\bigcup_{i \in N} A_i$

(ii) $A_i \cap [s]_i = \phi$, $(i \in N)$,

then $\bigcap_{i \in N} A_i \neq \phi$.

(II″) Let A_i, $(i \in N)$, be closed sets such that

(i) $[s]=\bigcup_{i \in N} A_i$

(ii) If $L (\neq \phi) \subset N$ and $J=N-L$, then $\bigcap_{i \in L} [s]_i \subset \bigcup_{j \in J} A_j$,

then $\bigcap_{i \in N} A_i \neq \phi$.

(III) [the Brouwer fixed point theorem]

If $F : [s] \longrightarrow [s]$ is continuous, then F has a fixed point, i. e., there is an $x^* \in [s]$ such that $F(x^*)=x^*$.

Remarks. About the domain and the range of the Brouwer fixed point theorem, we may take the one different from $[s]$. In fact, we take the domain and the range as the unit ball in the section 2, the compact convex set in R^n in the sectin 3 and the compact convex set in R^n_+ in the section 4, respectively. These are all homeomorphic. Therefore, the fixed point property is invariant.

Now, the following propositions holds.

Proposition. (I)$\Rightarrow$(II).

[proof] Let $K^0=[s]$ and K^r be the r-th subdivided complex from K^0. For any vertex c in K^r, there is a unique open carrier that contains c. Let this simplex be $(t)=(a^{i_0}, \ldots, a^{i_k})$. Then, by the assumtion of (II),

$$c \in (t) \subset [t] \subset A_{i_0} \cup \ldots \cup A_{i_k}.$$

For c, we define the labelling function $l : c \longmapsto N$ as follows. $l(c)$ may take any value among $\{i_0, \ldots, i_k\}$. Clealy, this mapping l is admissible.

Hence, there is a completely labelled simplex by (I). Let this simplex be $[c^{0_r}, \ldots, c^{n_r}]$. We can consider as $l(c^{i_r}) = i$ for some i, if necessary relabelling. Therefore $c^{i_r} \in A_i$. Also, we can assume that $\mu[K^r] \longrightarrow 0$ by appropriately subdividing.[1),2)]

Since $[s]$ is compact, we can assume that $c^{i_1}, \ldots, c^{i_r}, \ldots$ are converge. Let the limit of this sequence be c^*. Since A_i, $(i \in N)$, are the closed sets, $c^* = \lim_{r \to \infty} c^{i_r} \in A_i$, $(i \in N)$. Thus,

$$c^* \in \bigcap_{i \in N} A_i. \quad //$$

Proposition. (II) $\Rightarrow$ (III).

[proof] Let $x = \sum_{i=0}^{n} \lambda_i a^i$ be a point in $[s]$, where $\sum_{i=0}^{n} \lambda_i = 1$, $\lambda_i \geqq 0$, $(i \in N)$. Similarly, let $F(x) = \sum_{i=0}^{n} \lambda'_i a^i$, where $\sum_{i=0}^{n} \lambda'_i = 1$, $\lambda'_i \geqq 0$, $(i \in N)$. Then $A_i = \{x \mid \lambda'_i \leqq \lambda_i\}$ satisfy the conditions of (II), since

(i) they are closed, for the barycentric coordinate λ_i of x are continuous function of x;

(ii) If $x \in [a^{i_0}, \ldots, a^{i_k}]$, then $\lambda_{i_0} + \ldots + \lambda_{i_k} = 1$ and $\lambda'_{i_0} + \ldots + \lambda'_{i_k} \leqq 1$. Accordingly, there exists an index i_j, $(0 \leqq j \leqq k)$ such that $\lambda'_{i_j} \leqq \lambda_{i_j}$, and hence $x \in A_{i_j}$. Thus $[a^{i_0}, \ldots, a^{i_k}] \subset A_{i_0} \cup \ldots \cup A_{i_k}$.

According to (II), $\bigcap_{i \in N} A_i \neq \phi$. Let $x^* \in \bigcap_{i \in N} A_i$. Then $\lambda'_i \leqq \lambda_i$, $(i \in N)$ and

$$1 = \sum_{i=0}^{n} \lambda'_i \leqq \sum_{i=0}^{n} \lambda_i = 1.$$

Therefore, $\lambda'_i = \lambda_i$, $(i \in N)$, i. e., $F(x^*) = x^*$. //

Proposition. (III) $\Rightarrow$ (II′).

[proof] (Kawada [4]) Let A_i, $(i \in N)$, be closed sets such that satisfy conditions of (II′). Then, we must show that $\bigcap_{i \in N} A_i \neq \phi$. Suppose that $\bigcap_{i \in N} A_i = \phi$. Now, for any $x \in [s]$, we can define

$$\nu_i(x) = \frac{\rho(x, A_i)}{\sum_{i=0}^{n} \rho(x, A_i)}, \quad (i \in N)$$

where $\rho(x, A_i)$ is the distance between x and A_i. (By our assumption, $\sum_{i=0}^{n} \nu_i(x, A_i) > 0$.) Thus,

1) For example, the barycentric subdivision.

2) μ is the mesh of a complex K. The mesh is the maximum length among edges.

$$\begin{cases} 0 \leqq \nu_i(x) \leqq 1, & (i \in N) \\ \sum_{i=0}^{n} \nu_i(x) = 1 \end{cases}$$

i. e., $\{\nu_i\}$ is the partition of unitiy. For any $x \in [s]$, we correspond the point such that its barycentric coordinates with respect to $\{a^0, \ldots, a^n\}$ is $(\nu_1(x), \ldots, \nu_n(x))$. By this correspondence, we can define $F: [s] \longrightarrow [s]$. Clearly, F is continuous. Therefore, there is at least one fixed point x^*, i. e., $F(x^*) = x^*$ by (III). Then, x^* is included in some A_i, say A_j, because $\{A_i\}$, $(i \in N)$ is the closed covering by condition (i). If $x^* \in A_j$, then $\nu_j(x^*) = 0$ by definition. Hence, $x^* = F(x^*)$ is included in the face opposite to the xertex a^j i. e., $x^* \in [s]_j$. This is inconsistent with condition (ii). Therefore,

$\bigcap_{i \in N} A_i \neq \phi$. //

Proposition. (II′) ⇒ (I).

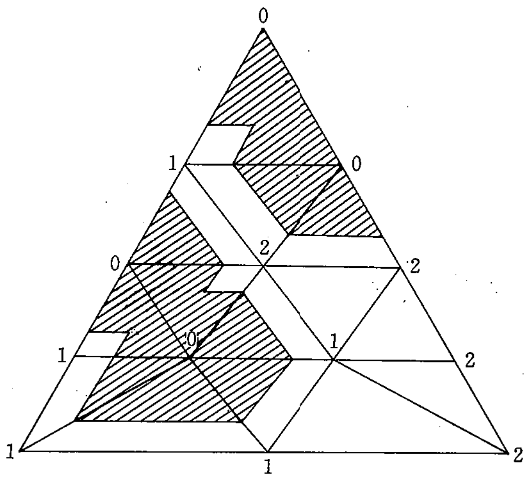
[proof][3] Let K be a triangulation of $[s]$. Consider any simplex $[t] \in K$. For each vertex c^i of $[t]$, we define $B_{i,[t]}$ as the set of points such that i-th coordinate is greater than $n/(n+1)$ in $[t]$, i. e.,

$$B_{i,[t]} = \left\{ x \mid x \in [t],\ \lambda_i \geqq \frac{n}{n+1} \right\}.$$

Furthermore, we define A_i as

$$A_i = \bigcup_{\Sigma_i} B_{i,[t]}$$

where $\sum_i$ runs over all the vertices that have label i in K.



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3) We can see the alternative proof of (III) ⇒ (I) in Yoseloft [12].

From its construction of A_i, every vertex is included in any one A_i. If the vertex c is included in A_i, then c has label i. Therefore, if there is a point in $\bigcap_{i \in N} A_i$, the carrier of that point is the completely labelled simplex.

Now, $\{A_i\}$, $(i \in N)$, is clearly the closed covering of $[s]$ and each A_i does not meet with $[s]_i$. Accordingly, $\{A_i\}$, $(i \in N)$, satisfy the conditions of (II′). Hence, there exists the point $x^* \in \bigcap_{i \in N} A_i$. Therefore, there is the completely lablled simplex. //

In Todd [10], we can see the proof of Propositions (I) ⇒ (II″) ⇒ (III).

After all, it was proved that the first group of propositions (I) ~ (III) are all equivalent. Simultaneously, it was also proved that K. K. M. theorem and its variants (II′), (II″) are equivalent.

2.

A subset A of a space X is called a retract of X, if there exists a continuous map $r: X \longrightarrow A$ such that $r|A = 1_A$. (1_A is an identity map in A.) And the map r is called a retraction of X onto A. When X is an arbitrary space and A a subset of X, then A is a retraction of X iff for every space Y each continuous map $h: A \longrightarrow Y$ has a continuous extention to X.

(IV) [No Retraction theorem]

S^{n-1} is not a retract of B^n.

Here, B^n is an unit ball in R^n and S^{n-1} is its sphere.

Let X and Y be two spaces and $I = [0, 1]$ the closed unit interval. Then, two maps $h_0, h_1 : X \longrightarrow Y$ are said to be homotopic, if there exists a continuous map $H : X \times I \longrightarrow Y$ with $H(x, 0) = h_0(x)$ and $H(x, 1) = h_1(x)$ for all $x \in X$. When h_0 and h_1 are homotopic, it is written as $h_0 \simeq h_1$. The map H is called a homotopy from h_0 to h_1 and is written as $H : h_0 \simeq h_1$. A continuous map that is homotopic to some constant map of X into Y is said to be nullhomotopic. A space X for which the identity map 1_X is nullhomotopic is said to be contractible.

A map $h : X \longrightarrow Y$ is called a homotopy equivalence, if there exist a $g : Y \longrightarrow X$ such that $g \circ h \simeq 1_X$ and $h \circ g \simeq 1_Y$. If there exists a homotopy equivalence $h : X \longrightarrow Y$, X and Y are said to be the same homotopy type and is written as $X \simeq Y$. A space X is contractible iff X has the same homotopy type as a point. It is expressed that a space is shrunk to a point.

(IV′) [Non-contractibility theorem]

S^{n-1} is not contractible.

Proposition. (III) ⇔ (IV) ⇔ (IV′).

[proof] By Theorem 2-28 and Theorem 3-6 in Naber [8] or Theorem 5.1 and Theorem 5.2 in Nikaido [9]. //

3.

Let's see the relation between the Brouwer fixed point theorem and Variatinal Inequalities.

(V) [Ky-Fan inequality] (Aubin [1])

Suppose that K is a compact convex set of a Banach space and that φ is a function $K \times K$ to R satisfying

(i) $^\forall y \in K$, $x \longmapsto \varphi(x, y)$ is lower semicontinuous

(ii) $^\forall x \in K$, $y \longmapsto \varphi(x, y)$ is concave,

then there exists $x^* \in K$ such that

$$\sup_{y \in K} \varphi(x^*, y) \leqq \sup_{y \in K} \varphi(y, y).$$

(V′) [Variational Inequalities] (Kinderlehrer and Stampacchia [5])

Let $K \subset R^n$ be compact convex set and let $F : K \longrightarrow (R)'$ be continuous. Then there is an $x^* \in K$ such that

$$<F(x^*), y - x^*> \geqq 0 \quad \text{for all } y \in K,$$

where $(R)'$ is the dual space of R^n and $< ., . >$ is the bilinear form.

(V″) $^\exists x^* \in K \subset R^n$,

$$(F(x^*), y - x) \geqq 0 \quad \text{for all } y \in K,$$

where $(.,.)$ is the inner product.

The following propositions hold.

Proposition. (III) ⇒ (V).

[proof] By Aubin [1], p. 203, Theorem 3. //

Proposition. (V) ⇒ (V′).

[proof] If we put $\varphi(x, y) = <F(x), x - y>$, then this satifies all the conditions of (V). Therefore, there exists $x^* \in K$ such that,

$$
\begin{aligned}
<F(x^*),\ x^*-y> &\leqq \sup\ \varphi(x^*,\ y) \\
&\leqq \sup\ \varphi(y,\ y) \\
&=0 \quad //
\end{aligned}
$$

Propositin. (V′)⇒(V″).

[proof] The inner product is the special case of bilinear form. Hence, (V″) is the corollary of (V′). //

Proposition. (V″)⇒(III).

[proof] If we define $\varphi(x, y)=(x-F(x), y-x)$, then this satisfies the conditions (V″), where $F(x)$ is assumed the function B^n to B^n. Therefore, by (V″) there exists $x^*\in B^n$ such that

$$(x^*-F(x^*),\ y-x^*)\geqq 0 \quad \text{for all } y\in B^n.$$

In particular, if we take $y=F(x^*)$, then

$$0\leqq (x^*-F(x^*),\ F(x^*)-x^*)=-||F(x^*)-x^*||$$

Hence, $F(x^*)=x^*$, i. e., there is a fixed point in B^n. //

4.

In the section 3, we saw that the existence of the solution for Variational Inequalities (V″) is equivalent to the Brouwer fixed point theorem (III). In that case, without loss of generality, we can restrict the domain to the compact convex set in R^n_+. In this section, we consider the problem in such restricted domain. Let's consider the following three problems.

Variational Inequalities Problem in R^n_+ :

find $x\in K\cap R^n_+$ such that

$$(F(x),\ y-x)\geqq 0 \quad \text{for all } y\in K\cap R^n_+,$$

where $F: K\cap R^n_+ \longrightarrow R^n$ is continuous map.

Nonlinear Complementarity Problem:

find $x\in K\cap R^n_+$ such that

$$(F(x),\ x)=0,$$

subject to $F(x)\geqq 0$, where $F: R^n_+\longrightarrow R^n_+$ is continuous map.

The Problem of Walrasian equilibrium:

find $x\in[u]$ (the unit simplex; the space of prices) such that

$$F(x)\geqq 0$$

subject to $(F(x), x) = 0$ (Walras law),

where $F : [u] \longrightarrow R^n$ is continuous map (excess supply function).

The assertion that there exists a solution for Nonlinear Complementarity Problem can be expressed as follows.

(VI) $\exists x^* \in K \cap R^n_+$,
$F(x^*) \geqq 0$
$(F(x^*), x^*) \geqq 0.$

Then, we can ascertain the following proposition.

Proposition. (VI) $\Rightarrow$ (V″).

[proof] Self-evident. //

Next, the assertion that there exists a solution for the Problem of Walrasian equilibrium can be expressed as follows.

(VII) $\exists x^* \in [u]$,
$F(x^*) \geqq 0$
$(F(x^*), x^*) = 0$

Now, we can verify the following proposition.

Proposition. (VII) $\Rightarrow$ (VI).

[proof] We can consider Nonlinear Complementarity Problem for the normalized variables. Then (VI) can be reduced to (VII). Therefore (VI) follows from (VII). //

At last, we consider the following proposition.

Proposition. (V″) $\Rightarrow$ (VII).

[proof] For $x \in [u]$, satisfying $(F(x), x) = 0$, apply (V″). Then, there is $x^* \in [u]$, for which

$$(F(x^*), y) \geqq 0 \quad \text{for all } y \in [u]$$

holds. In particular, if we select $y = e_i$ (unit vector), then

$$f_i(x^*) \geqq 0.$$

Therefore, $F(x^*) = (f_1(x^*), \ldots, f_n(x^*)) \geqq 0$. Hence, we get the result of (VII). //

Remark. By this process of proof, we got the alternative proof of the equivalence between (III) and (VII) indirectly.

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