ON THE PORTFOLIO SELECTION CRITERIA USING STOCHASTIC DOMINANCE

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I. Introduction

An investor is assumed to have a complete preference order among the set of all available portfolios. If we identify each portfolio as the distribution function of its return, $F$, and the investor is characterized by his utility function, $u$, then his preference order is expressed by the value of expected utility, $E(u)$.

In an economic study, however, we are not interested in the preference of a particular individual, but in the characters of collective of individuals having only certain universal properties in common (e.g. risk aversion). We denote such a collective as $\{u\}$, and the set of all available portfolios as $\{F\}$.

For a given $\{u\}$, $F$ dominates $G$ ($F \triangleright G$) means $E_F(u) \geq E_G(u)$ for all $u \in \{u\}$, and $E_F(u) > E_G(u)$ for some $u \in \{u\}$. The relation $\triangleright$ gives the partial ordering on $\{F\}$. Hanoch and Levy, in their excellent article (1969), presented the necessary and sufficient conditions for $F \triangleright G$, for two typical $\{u\}$'s, introducing the concept of stochastic dominance. Such a criterion as $\triangleright$ which gives the partial order on $\{F\}$ constitutes a subset of $\{F\}$, called the efficient set; any other element is never preferred to an element of the efficient set by the criterion. Furthermore, Hanoch and Levy pointed out that the traditional mean-variance criterion for obtaining an efficient set contradicts the expected utility theory without proper restrictions on $\{u\}$ and/or $\{F\}$, and they presented some of these restrictions. The next section is a brief review of their results.

Using the results of Hanoch and Levy, Philippatos and Gressis (1975) discussed the equivalence conditions between the efficient sets by stochastic dominance criterion and that of mean-variance criterion in the case of uniform distribution, normal distribution, and lognormal distribution. In this paper, focusing on the same distributions as

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Philippatos and Gressis, the relation between the two criteria is made clear using the graphical representation of the set of distributions on the $(\mu, \sigma)$ plane which dominate and are dominated by a given distribution. It will be revealed that particularly, in the case of lognormal distribution, the "mean–coefficient of variation" criterion coincides with the stochastic dominance criterion.

II. Review of Stochastic Dominance

Hanoch and Levy's theorems are reviewed briefly without proof in this section.

Theorem 1. Let $\{u\}$ be the set of all non-decreasing functions with finite value in finite intervals. $F$ and $G$ may be any cumulative distribution.

$$F \triangleright G \Leftrightarrow G(x) - F(x) \geq 0,$$

for every $x$, strict inequality holding for some $x$. This condition for $F$ and $G$ is called the first order stochastic dominance criterion ($FSD$).

Theorem 2. Let $\{u\}$ be the set of all non-decreasing and concave functions (property of risk averter). $F$ and $G$ are as in theorem 1.

$$F \triangleright G \Leftrightarrow \int_{-\infty}^{\infty} (G(t) - F(t)) dt \geq 0,$$

for every $x$, strict inequality holding for some $x$. This condition for $F$ and $G$ is called the second order stochastic dominance criterion ($SSD$). It is evident that $F \triangleright G$ by $FSD$ criterion leads $F \triangleright G$ by $SSD$ criterion.

Theorem 3. This theorem gives the simplest criterion for $F \triangleright G$ when $F$ and $G$ intersect only once, $F$ crossing $G$ from below. $\{u\}$ is as in the theorem 2.

$$F \triangleright G \Leftrightarrow \mu_1 \geq \mu_2,$$

$\mu_1$ and $\mu_2$ is the mean value of $F$ and $G$ respectively.

Theorem 4. This theorem represents a restriction on $\{u\}$ for attaining the equivalence between the $SSD$ criterion and the mean-variance criterion. The restriction on $\{u\}$ is that of theorem 2. The restriction on $\{F\}$ is that they are the same type of distributions, that is, the differences
among them are only the values of centerizing and scale parameters.

III. Examination in the Case of Special Distributions

The SSD and mean-variance criteria are applied to the families of distributions with two parameters which are the independent functions of mean, \( \mu \), and variance, \( \sigma^2 \). A distribution of such a family is represented as a point on the \((\mu, \sigma)\) plane, and from here on we identify a distribution as a point on the plane. The plane can be divided into three parts for the fixed given distribution \( F_0 \): \([F|F_{D}F_0]\), \([F|F_{D}F\]_, and the remaining region. A comparison between the two criteria can be undertaken by examining the shape of the regions. It should be noted that, in the case when the shape of \([F|F_{D}F_0]\) on the \((\mu, \sigma)\) plane does not vary by the change of \( F_0 \), the other two regions are determined by the shape of \([F|F_{D}F\]_, this is because \([F|F_{D}F]\\text{ and }[F|F_{D}F_0]\) are symmetric, with \( F_0 \) as the point of symmetry; let \( F_1 \) and \( F'_1 \) be two symmetric points with \( F_0 \) as the point of symmetry, then it can be easily seen that

\[ F_1 \in [F|F_{D}F_0] \iff F'_1 \in [F|F_{D}F]. \]

Uniform Distribution on \((a, b)\). In this case, \( \mu = (b + a)/2 \), \( \sigma = (b - a)/2\sqrt{3} \), and hence \( a = \mu - \sqrt{3}\sigma \), \( b = \mu + \sqrt{3}\sigma \). We take \( F_0 \) as an uniform distribution with parameters \( a_0 \), \( b_0 \) (\( \mu = \mu_0 \), \( \sigma = \sigma_0 \)). Applying theorem 1, \( a \leq a_0 \), and \( b \leq b_0 \) (at least one strict inequality holds) yields \( F_{D}F \). In the case

\[ \begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{Figure 1.}
\end{figure} \]
of $a < a_0$ and $b > b_0$, since the assumptions of theorem 3 hold, $\mu < \mu_0$ yields $F_0F$. These conditions for $(a, b)$ determine a region $\{\mu, \sigma | F_0F\}$, (the shaded area in Figure 1). Since the shape of the region is invariant regardless of changes of $(\mu, \sigma)$, we immediately obtain a region where $F_0$ and $F$ are efficient, (the dotted area in Figure 1).

Normal Distribution. Since the assumption of theorem 4 holds (the same type of distribution), there is no problem to examine.

Before proceeding to the next case, let us consider the families of distributions which are transformed to a set of straight lines by certain continuous transformations of both $a$ and $b$. It is evident that two distinct distributions of these family intersect at most once, and theorem 3 can be applied. The following theorem gives the general expression for these family.

Theorem. For a continuous strictly increasing function $\phi$, $R \rightarrow R$ and another continuous strictly increasing function $\varphi$, $0, 1 \rightarrow R$ (that is $\varphi^{-1}$ is a distribution function), and constants $\alpha, \beta$,

$$\varphi F\phi(t) = (t-\alpha)/\beta \iff F(x) = \varphi^{-1}((\varphi^{-1}(x) - \alpha)/\beta)$$

The set of such distributions for fixed $\phi$ and $\varphi$ constitutes a family of distributions with parameters $\alpha, \beta$, and we denote each element as $F_{a, b}$. Without loss of generality, we can choose $\phi$ and $\varphi$ as $F_{0, 1}\phi$ having mean 0 and variance 1 (providing variance exists). Since $\varphi F_{0, 1}\phi = i$, $\varphi^{-1} = F_{0, 1}\phi$,

$$\{F_{a, b}\} \equiv \{F_{0, 1}\phi((t-\alpha)/\beta)\}$$

constitutes a family such that each member differs only by values of the centering parameter, $\alpha$, and the scale parameter, $\beta$.

proof ($\Rightarrow$) Operating $\varphi^{-1}$ to the left,$$
F\phi(t) = \varphi^{-1}((t-\alpha)/\beta),
\therefore F(x) = \varphi^{-1}((\varphi^{-1}(x) - \alpha)/\beta), \text{ where } x = \phi(t)
$$

($\Leftarrow$) Operating $\varphi$ to the right,$$
\varphi F(x) = (\varphi^{-1}(x) - \alpha)/\beta,
\therefore \varphi F\phi(t) = (\varphi^{-1}(t) - \alpha)/\beta = (t-\alpha)/\beta
$$

It is evident that $F_{a, b}$ crosses $F_{a', b'}$ from below if and only if $\beta' > \beta$. Applying theorem 3, we obtain the dominance criterion.
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$\beta' > \beta$, $\mu' < \mu \Leftrightarrow F_{\alpha, \beta} \leq F_{\alpha', \beta'}$,  
where $\mu$ and $\mu'$ are the mean values of $F_{\alpha, \beta}$ and $F_{\alpha', \beta'}$.

Lognormal distribution with parameters, $m$, $s$. Letting $\varphi^{-1}$ be $F_{\alpha, \phi}(x) = \Phi((\log x - m)/s)$. In this case, $\mu = \exp (m + s^2/2)$, $\sigma = \mu \exp s^2 - 1$, then we obtain $s^2 = \log ((\mu/\sigma)^2 + 1)$, $m = \log (\mu/\sqrt{(\sigma/\mu)^2 + 1})$. Since the scale parameter of $F_{\alpha, \phi}$ is $s$, $s > s_0$, $\mu < \mu \Leftrightarrow F_{\alpha', \phi} < F_{\alpha, \phi}$. It can be easily seen that $s > s_0$ is equivalent to $\sigma/\mu > \sigma_0/\mu_0$ (Figure 2).

IV. Concluding Remarks

Figure 1 and 2 show that in the case of uniform distribution and lognormal distribution, the mean-variance criterion is a sufficient condition for the SSD, but not a necessary condition. In the case of lognormal distribution, the use of coefficient of variation in place of variance results in the coincidence with the SSD criterion.

References


1. For the empirical study, Porter and Gaumnitz (1972) have concluded that the difference between the two criteria is not critical except for the highly risk-averse investor.

