

OPTIMAL TERMINAL CONTROL OF LINEAR DYNAMICAL SYSTEMS WITH AMPLITUDE CONSTRAINTS AND CONVEX CRITERION FUNCTION**

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I Introduction

The terminal control problem which will be dealt with in this paper is to find an optimal control which minimizes the value of a given criterion function that is defined as a continuously differentiable convex function of the terminal state. The system to be considered is a linear differential system (1) and constraints to which the control is subject are the amplitude (or magnitude) constraints (2). The purpose of this paper is to give a new numerical solution method for the optimal terminal control problem.

The problem under consideration has been dealt with theoretically and computationally in various ways; the application of Maximum Principle [1], [2], the application of Dynamic Programming [3], [4], the method of Functional Analysis [5], the application of hill-climbing method [6], and so forth. As for the systems with the constraints under consideration, however, Ho's paper [6] has been the only one in which an exact computational algorithm was proposed for the quadratic criterion function, and there still remains an urgent need to further develop numerical solution methods.

In this paper the problem will be formulated as a kind of nonlinear programming problem, in which the objective function is the criterion function, which is assumed continuously differentiable and convex, and the constraint set is the set of all reachable terminal states. The analytical determination of the set of reachable states is quite difficult to achieve in general and it is hard to apply

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usual nonlinear programming techniques; Gradient Projection Method [7], [8], Method of Feasible Directions [9], Cutting Plane Method [10] and so on [11], [12]. For the purpose of filling this gap, a new nonlinear (convex) programming method will be proposed in this paper, which is successfully applicable to the problem. Based on this convex programming method, an efficient computational algorithm will be presented with an example by way of demonstration.

II Formulation of the problem

It is assumed in this paper that the dynamics of a linear continuous system is described by the following state equation:

$$(1) \quad dx(t)/dt = A(t)x(t) + B(t)u(t),$$

where x is the n -dimensional column state vector, u is the r -dimensional column control vector, and $A(t)$ and $B(t)$ are $n \times n$ and $n \times r$ continuous matrices respectively. Each component $u_i(t)$ of the control vector $u(t)$ is assumed to be a Lebesgue-measurable function of t and subject to the so-called amplitude constraints

$$(2) \quad |u_i(t)| \leq 1 \text{ for } i=1, 2, \dots, r.$$

Given the initial time t_0 , an initial state $x(t_0)$ and the terminal time, $t_f(>t_0)$, the terminal state $x(t_f, u)$ may be calculated as follows:

$$(3) \quad x(t_f, u) = \Phi(t_f)x(t_0) + \Phi(t_f) \int_{t_0}^{t_f} \Phi^{-1}(\tau)B(\tau)u(\tau)d\tau,$$

where $\Phi(t)$, the $n \times n$ fundamental matrix, is the solution of the following matrix differential equation,

$$(4) \quad d\Phi(t)/dt = A(t)\Phi(t), \quad \Phi(t_0) = E \quad (n \times n \text{ identity matrix}).$$

The set of all measurable vector functions u which satisfy the amplitude constraints (2) is denoted by U . Then, due to the result of LaSalle [13] it is realized that the set of all possible terminal states $x(t_f, u)$, denoted by S , forms a compact (bounded closed) convex set in the n -dimensional Euclidean space E_n when u runs through the set of admissible controls U .

Let $f(x)$ be a given continuously differentiable convex function defined in an open set including the set S . The terminal control problem which is to be considered in this paper is to find an optimal control u which minimizes the value of the given function $f(x(t_f, u))$ under the amplitude constraints (2).

If the function f is linear, an optimal solution can be obtained in a closed-form, [13]. To show this, let $f(x)$ be the linear function defined by

$$(5) \quad f(x) = (\eta, x),$$

where η is a n -dimensional constant vector and the symbol (η, x) denotes the usual inner product of the vectors η and x . Then, by the use of equation (3) the value $f(x(t_f, u))$ can be calculated as follows;

$$(6) \quad f(x(t_f, u)) = \eta^T x(t_f, u) \\ = \eta^T \Phi(t_f) x(t_0) + \int_{t_0}^{t_f} [\eta^T \Phi(t_f) \Phi^{-1}(\tau) B(\tau)] u(\tau) d\tau,$$

where the superscript T means transpose.

The first term on the right-hand side, $\eta^T \Phi(t_f) x(t_0)$, is a constant independent of the choice of u . Therefore, it is sufficient to minimize the second term for $u \in U$. For notational simplicity a vector function $v(\tau)$ is introduced here,

$$(7) \quad v(\tau) = (v_1(\tau), v_2(\tau), \dots, v_r(\tau)) \\ = \eta^T \Phi(t_f) \Phi^{-1}(\tau) B(\tau).$$

Then the second term of the right-hand side of equation (6) may be rewritten in the following way:

$$(8) \quad \int_{t_0}^{t_f} v(\tau) u(\tau) d\tau = \int_{t_0}^{t_f} \sum_{i=1}^r [v_i(\tau) u_i(\tau)] d\tau.$$

Now, it is obvious that an optimal solution is given by the closed form,

$$(9) \quad u_i^*(\tau) = -\text{sgn}[v_i(\tau)] \text{ for } t_0 \leq \tau \leq t_f \text{ and } i=1, 2, \dots, r,$$

where $\text{sgn}[v_i] = 1$ if $v_i > 0$, $= -1$ if $v_i < 0$, and $= 0$ if otherwise.

Hence the point

$$(10) \quad x(t_f, u^*) = \Phi(t_f) x(t_0) - \int_{t_0}^{t_f} \Phi(t_f) \Phi^{-1}(\tau) B(\tau) \text{sgn}[v(\tau)] d\tau$$

gives the minimum value of the function $f(x)$ on S ,

$$(11) \quad \text{Min}_{x \in S} f(x) = \eta^T \Phi(t_f) x(t_0) - \int_{t_0}^{t_f} \sum_{i=1}^r |v_i(\tau)| d\tau.$$

The criterion function $f(x)$, in general, is nonlinear, and it cannot be expected to find an optimal control in a closed form. Since the set S is compact and $f(x)$ is continuous, the existence of the minimum of the function f on the set S is evident. Moreover, according to the above paragraph, linear programming, i.e., to find a minimum of a linear form on the set S , can be solved analytically.

Therefore the terminal control problem under consideration can be easily solved if there is available a method of convex programming solely based on the feasibility of linear programming. Such a method of non-linear programming for a general convex objective function has not been developed so far. For the purpose of filling this existing gap the present author will propose a new method of convex programming in the following section.

III A new method of convex programming

Summarizing the problem, the constraint set within which the n -dimensional variable x has to lie is a compact convex set S in the n -dimensional Euclidean space E_n . The objective function to be minimized, $f(x)=f(x_1, x_2, \dots, x_n)$, is a continuously differentiable convex function defined in an open set including the set S . Linear programming problems can be solved, that is, a minimum of any linear form (η, x) can be obtained analytically, where η is a n -dimensional constant non-zero vector. The problem which is to be considered in this section is to find the minimum of the objective function $f(x)$, that is, to find an optimal point $x^* \in S$ so that $f(x^*) \leq f(x)$ for any $x \in S$.

As has been discussed in the preceding section it is evident there exists an optimal point $x^* \in S$.

If, in addition, f is "strictly" convex, the uniqueness of the optimal point x^* is clear because of the convexity of the constraint set S . To show this suppose there are two distinct optimal points x^* and x^{**} in S . Assuming that f is strictly convex it follows that for any real number λ ($0 < \lambda < 1$),

$$f[\lambda x^* + (1-\lambda)x^{**}] < \lambda f(x^*) + (1-\lambda)f(x^{**}) = f(x^*) = \underset{x \in S}{\text{Min}} f(x).$$

Since S is convex, the points $\lambda x^* + (1-\lambda)x^{**}$ belong to S , which is the desired contradiction. If f is not "strictly" convex, uniqueness of the optimal point can not be guaranteed.

Now, a well known inequality is introduced for the purpose of repeated use. [14]. For any two distinct points x and y in S , the following inequality holds because of the convexity of f ,

$$f((1-\lambda)x + \lambda y) \leq (1-\lambda)f(x) + \lambda f(y) \quad \text{for } 0 < \lambda < 1.$$

This may be rewritten as follows:

$$f(y) - f(x) \geq \|y - x\| \cdot \frac{1}{\lambda \|y - x\|} \{f(x + \lambda(y - x)) - f(x)\},$$

where the symbol $\|$ stands for the Euclidean norm. Letting λ tend to zero, the right-hand side of the above inequality tends to $\|y-x\| \lim_{\lambda \rightarrow 0} \{f(x + \lambda(y-x)) - f(x)\} / \lambda \|y-x\|$. The limit of the term under the limit symbol is the directional derivative of f at the point x along the direction vector $y-x$, which is given by $(\nabla f(x), y-x) / \|y-x\|$. Therefore for any x and y in S ,

$$(12) \quad f(y) - f(x) \geq (\nabla f(x), y-x).$$

An iterative procedure is given below to generate a sequence of points $\{x^k\}$ which converges to the uniquely determined optimal point x^* provided that the objective function f is "strictly" convex. The sequence of real numbers $\{f(x^k)\}$ is monotone decreasing and converges to the minimum value even if the function f is not "strictly" convex.

At the outset of the procedure, one may choose as a starting point $x^0 \in S$ arbitrarily. One cycle of the iteration is as follows: At the k -th stage $x^k \in S$ is given,

Step 1: Compute the gradient of function $f(x)$ at the point

$$x^k \in S, \text{ that is, } (\nabla f(x^k))^T = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n} \right) \Big|_{x=x^k}.$$

If the gradient is the null vector, then it is clear that $f(x^k)$ is the minimum [11], [15] and the iteration terminates. Otherwise go to Step 2.

Step 2: Minimize the linear form $(\nabla f(x^k), x)$ on S , that is, find a point y^k such that $\min_{x \in S} (\nabla f(x^k), x) = (\nabla f(x^k), y^k)$. If $(\nabla f(x^k), y^k) = (\nabla f(x^k), x^k)$, then $f(x^k)$ is the minimum and the iteration ceases here. Otherwise, i.e., if

$$(13) \quad (\nabla f(x^k), y^k) < (\nabla f(x^k), x^k),$$

minimize $f(\theta y^k + (1-\theta)x^k)$ under the constraint $0 \leq \theta \leq 1$ and denote the optimal value of θ by θ_k . Then calculate $x^{k+1} = \theta_k y^k + (1-\theta_k)x^k$ and go back to Step 1 of the following cycle.

The algorithm mentioned above needs a few comments. The determination of the point y^k is feasible because of the feasibility assumption of linear programming. The minimization of $f((1-\theta)x^k + \theta y^k)$ is the one-dimensional minimization problem and its feasibility is implicitly assumed here. Now assume that $(\nabla f(x^k), x^k) = (\nabla f(x^k), y^k) \leq (\nabla f(x^k), x)$ for any $x \in S$, then $(\nabla f(x^k), x - x^k) \geq 0$, which implies $f(x) \geq f(x^k)$ for any $x \in S$ due to the inequality (12). Therefore the iteration has been designed to cease in this case.

First shown is the monotonicity of the sequence $\{f(x^k)\}$, that is

$f(x^{n+1}) < f(x^n)$, provided that the iterative procedure does not terminate at the n -th cycle. It is clear that the following inequality holds for $0 \leq \theta \leq 1$;
 $f(x^{n+1}) \leq f((1-\theta)x^n + \theta y^n) = f(x^n + \theta(y^n - x^n))$. Therefore it suffices to show that the derivative of the right-hand side with respect to θ at $\theta=0$ is negative. The derivative can be evaluated in the following way as in the case of the derivation of (12):

$$(14) \quad \frac{d}{d\theta} f(x^n + \theta(y^n - x^n)) = (\nabla f(x^n), y^n - x^n),$$

which is negative because of the inequality (13).

Next shown is the convergence of $\{f(x^n)\}$ to the minimum $f(x^*)$ provided that the iterative procedure does not terminate in a finite number of steps. Since the iteration does not terminate, the inequality (13) is satisfied for any $k=0, 1, 2, \dots$. Also the following two inequalities are satisfied for any $k=0, 1, 2, \dots$;

$$(15) \quad (\nabla f(x^k), y^k - x^*) \leq 0,$$

and

$$(16) \quad (\nabla f(x^k), x^* - x^k) < 0.$$

The first inequality is evident and the second one follows from the fact that $(\nabla f(x^k), x^* - x^k) \geq 0$ implies $f(x^*) \geq f(x^k)$ according to the inequality (12). Using the inequality (15) it follows

$$(17) \quad (\nabla f(x^k), x^k - y^k) = (\nabla f(x^k), x^k - x^*) + (\nabla f(x^k), x^* - y^k) \geq (\nabla f(x^k), x^k - x^*).$$

Also the use of the inequality (12) yields

$$(18) \quad (\nabla f(x^k), x^k - x^*) \geq f(x^k) - f(x^*).$$

Combining these two inequalities (17) and (18) there holds

$$(19) \quad (\nabla f(x^k), x^k - y^k) \geq f(x^k) - f(x^*) \text{ for } k=0, 1, 2, \dots$$

Now assume that there is a positive number K so that $f(x^k) - f(x^*) \geq K > 0$. This assumption implies the following;

$$(20) \quad (\nabla f(x^k), x^k - y^k) \geq K.$$

Using the mean-value theorem, there follows for $0 \leq \theta \leq 1$

$$(21) \quad f(x^{k+1}) \leq f(x^k + \theta(y^k - x^k))$$

$$\begin{aligned}
&= f(x^k) + \theta \left[-\frac{d}{d\theta} f(x^k + \theta(y^k - x^k)) \right]_{\theta=\lambda(\theta)} \\
&= f(x^k) + \theta(\nabla f(\xi), y^k - x^k),
\end{aligned}$$

where $\xi = x^k + \lambda(\theta)\theta(y^k - x^k)$ for $0 \leq \lambda(\theta) \leq 1$.

Hence there follows

$$\begin{aligned}
(22) \quad f(x^k) - f(x^{k+1}) &\geq \theta(\nabla f(\xi), x^k - y^k) \\
&= \theta[(\nabla f(x^k), x^k - y^k) + (\nabla f(\xi) - \nabla f(x^k), x^k - y^k)] \\
&\geq \theta[K + (\nabla f(\xi) - \nabla f(x^k), x^k - y^k)].
\end{aligned}$$

Since the set S is bounded, there is a positive number M such that $\|x^k - y^k\| \leq M$. Since the gradient of $f(x)$ is continuous, it is uniformly continuous on the compact set S . Hence there exists a positive number $\delta > 0$ such that $\|\eta_1 - \eta_2\| \leq \delta$ and $\eta_1, \eta_2 \in S$ imply $\|\nabla f(\eta_1) - \nabla f(\eta_2)\| \leq K/2M$. Therefore choosing $\theta = \bar{\theta} = \delta/M$, $\|\xi - x^k\| \leq \delta$ and $\|\nabla f(\xi) - \nabla f(x^k)\| \leq K/2M$. Using the Schwarz inequality the following estimation is valid

$$|(\nabla f(\xi) - \nabla f(x^k), y^k - x^k)| \leq K/2.$$

Combining this with the relation (22), one obtains

$$f(x^k) - f(x^{k+1}) \geq \bar{\theta}K/2 = \delta K/2M > 0.$$

The constant $\delta K/2M$ is a constant independent of k , and therefore the above inequality implies $f(x^k) \rightarrow -\infty$ as $k \rightarrow \infty$, which is the desired contradiction. Thus, the convergence of $\{f(x^k)\}$ to the minimum $f(x^*)$ has been demonstrated. It is quite easy to see that this implies the convergence of $\{x^k\}$ to x^* whenever the minimum point x^* is uniquely determined.

It has to be noted that the error $f(x^k) - f(x^*)$ is not greater than the difference $(\nabla f(x^k), x^k) - (\nabla f(x^k), y^k)$ due to the inequality (19) when truncating the iterative procedure at the k -th cycle. Therefore, the value of the minimum can be obtained within any preassigned accuracy.

In conclusion the characteristic feature of the new method of convex programming described above has to be made clear. The main feature is that the direction $(y^k - x^k)$ along which the point traverses is determined globally. That is to say that the total structure of the constraint set S determines the direction but the local structure in the neighborhood of the point x^k does not. This can be best seen by comparing this method with, for instance, the gradient projection method [7], [8] in which the local structure mainly effects the determination of

the direction when the point x^k is a boundary point of S . According to this, the new method may be called the gradient projection method from the broader point of view. Other characteristic features are as follows; it is not necessary to know whether the point x^k is a boundary point of S or not and the geometry of S can be arbitrarily given as long as linear programming is feasible. These characteristic features built into this method make possible its successful application to the terminal control problem as will be shown in the following section.

IV Computational algorithm and illustrative example

The new computational method for convex programming obtained in the preceding section can be used to provide a computational algorithm for solving the optimal terminal control problem described in section 2.

To begin the iteration one may choose any control u^0 which satisfies the constraint (2) and compute the corresponding terminal state from the equation (3), which is denoted by x^0 . The pair (x^0, u^0) provides the point with which iteration starts. At the k -th stage pair (x^k, u^k) is given, and one cycle of iteration is as follows;

Step 1. Compute the gradient of function $f(x)$ at the point $x^k \in S$ that is, $(\nabla f(x^k))^T = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n} \right) \Big|_{x=x^k}$. If this vector is null, then the pair (x^k, u^k) gives the desired minimum and the iteration terminates. Otherwise go to Step 2.

Step 2. Minimize the linear form $(\nabla f(x^k), x)$ on S , that is, find a terminal state y^k such that $\text{Min}_{x \in S} (\nabla f(x^k), x) = (\nabla f(x^k), y^k)$. The terminal state y^k is determined by the following control $w^k = -\text{sgn}[\nabla f(x^k)\Phi(t_f)\Phi^T(\tau)B(\tau)]$. If $(\nabla f(x^k), y^k) = (\nabla f(x^k), x^k)$, then the pair (x^k, u^k) gives the desired minimum and the iteration ceases here. Otherwise, i.e., if $(\nabla f(x^k), y^k) < (\nabla f(x^k), x^k)$, minimize $f(\theta y^k + (1-\theta)x^k)$ under the constraint $0 \leq \theta \leq 1$. Let θ_k be the minimizing value of θ , then calculate $x^{k+1} = \theta_k y^k + (1-\theta_k)x^k$ and $u^{k+1} = \theta_k w^k + (1-\theta_k)u^k$.

Now shown is a simple numerical example. The state equation and the constraint are as follows;

$$\begin{pmatrix} dx_1(t)/dt \\ dx_2(t)/dt \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} 3 \\ 1 \end{pmatrix} u(t),$$

and

$$|u(t)| \leq 1 \text{ for } 0 \leq t \leq 1.$$

Given the initial time, $t_0=0$, an initial state, $x(t_0)=0$ and the terminal time, $t_f=1$, the terminal state x can be computed as follows;

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 \int_0^1 u(\tau) d\tau \\ e \int_0^1 e^{-\tau} u(\tau) d\tau \end{pmatrix}.$$

The criterion function to minimize is quadratic and of the form

$$f(x) = 2x_1^2 - 2x_1x_2 + 2x_2^2 - 6x_1 + x_2.$$

The numerical results obtained with the aid of a digital computer are shown in Table 1. In this case, obtained data indicate the minimum value

Table 1. Results of Iterations.

No. of Cycles.	x_1	x_2	y_1	y_2	$f(x)$	θ
0*	0.00000	0.00000	3.00000	1.71828	0.00000	0.59880
1	1.79640	1.02891	0.24795	-0.28172	-1.02891	0.07733
2	1.67665	0.92755	2.59117	1.36018	-4.89970	0.18447
3	1.84535	1.00736	-0.89939	-0.87986	-4.94243	0.03331
4	1.75392	0.94449	1.34417	0.40717	-4.95555	0.34325
5	1.61327	0.76006	3.00000	1.71828	-5.01127	0.11571
6	1.77373	0.87093	1.37396	0.42770	-5.05176	0.21827
7	1.68647	0.77419	3.00000	1.71828	-5.06883	0.06733
8	1.77491	0.83776	1.61019	0.59419	-5.08131	0.54107
9	1.68579	0.70597	3.00000	1.71828	-5.10844	0.07405
10	1.78310	0.78093	2.02565	0.90339	-5.12403	0.02034
11	1.83243	0.80584	-0.50915	-0.68911	-5.12768	0.01036
12	1.80818	0.79036	0.89927	0.11233	-5.12858	0.01989
13	1.79010	0.77687	1.83327	0.75755	-5.12911	1.00000
14	1.83327	0.75755	-0.56048	-0.71492	-5.15014	0.00570
15	1.81962	0.74915	0.67613	-0.02751	-5.15042	0.00640
16	1.81230	0.74418	1.27067	0.35695	-5.15050	0.00730
17	1.80835	0.74136	1.57844	0.57144	-5.15053	0.01134
18**	1.80574	0.73943			-5.15054	

* The starting point corresponds to the control function $u(t)=0$ ($0 \leq t \leq 1$).

** The optimal terminal state, and minimum value of $f(x)$ which is accurate within the error size 0.001.

$f(x)=-5.15054$, of which error was expected to be less than 0.001, the optimal terminal state $x=(1.80574, 0.73943)$ after eighteen iterations, and the derived optimal control is shown in Figure 1.

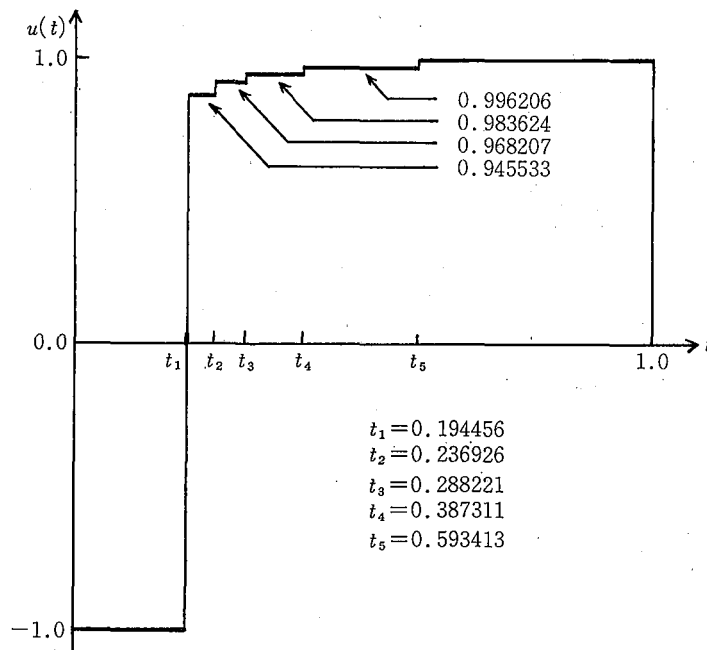


Figure 1. Optimal control function of the example.

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