

Temporal General Equilibrium in Overlapping Generations Model with Self-accumulated Pension

Kazushi Nishimura

Abstract: We shall aim for the temporary general equilibrium theory in spot and forward perishable goods markets and spot and forward bond markets for life period. In this model, each spot market and each forward market are segmented and so bond markets. Flow markets of goods and stock ones of bond are also segmented. After each working consumer withdraws fixed fiat money and each retired redeems bond, equilibrium prices are decided in the goods spot and forward markets. After each working invests his withdrawn money on bonds and each retired replace his bond, equilibrium prices are decided in bond spot and forward markets.

Keyword: Pension, Term Structure, Overlapping Generations, Temporary General Equilibrium

JEL: C61, D53, D84, E43

1. Introduction

We consider that in the competitive market, all the participants behave themselves as completely selfish and self-supported.

Samuelson[18] assumed that men enter the labor market at about the age of twenty and they work for forty-five years or so and then live for fifteen years in retirement. We are interesting in the working generations and the retired generations who are completely selfish and self-supported.

In a model, the working consumers have endowment till their retirement and in each period withdraw fixed money from his income and accumulate his own fund with government bond of v term periods. The retired consumers receive redemption of bond and its coupons as self-accumulated pension and buy discount bond of one period. There exists no bequest. The newcomers in the market have τ periods till his retirement and the new retired ones have the last v life-span. We consider the temporal general equilibrium in an exchange economy. This type of overlapping generations model were developed in 1970's by Grandmont and Laroque[9], Grandmont and Younes[10, 11] and Grandmont and Hildenbrand[6]. In the paper[16], we exercised two period model of Grandmont and Hildenbrand under the uncertainty. Here, we extend it to a multi-period model with age structure in goods spot and forward markets and bond spot and forward markets.

Consumption loan is possible if a consumer borrows money only for one period. However, a retired consumer cannot borrow any money because of the assumption of no endowment. Since we

are interesting in the existence of equilibrium in the market with private pension, we shall focus on the function of the fund without banking. On behalf of no borrowing and lending, there is no interest for money hoarding.

2. Definitions and Assumptions in Goods Spot and Forward Markets

We consider an exchange economy, where trade starts at period $t = 1$ and continues sequentially over time. A newcomer who initially enters into a market works for τ periods and lives for v periods after his retirement. Each consumer saves his or her own fund for the retirement life period with no bequest. Each generation is denoted by an age index $h = 1, 2, \dots, \tau, \dots, \tau + v$. If $h = 1$, the generation is a newcomer in the economy. If $h = \tau + 1$, the generation is retired. If $h = \tau + v$, the lifetime is over. The age structure in the society has been stationary and therefore let the population of each generation the same and constant.

At each period there exist l perishable goods, paper money, a discount bond to have a maturity of one period and government bond to have a maturity of v periods. A consumer who works for τ periods is endowed with ${}^h e \in R_+^l$ which is perfectly foresighted, where R_+^l is a closed positive cone of R^l . At period $t = 1$, every consumer has no debt. The retired are not endowed, but withdraw their fund ${}^h d_1$, which has been accumulated for the working period. For simplicity, all the consumers cannot borrow and lend money from the others. We assume

Assumption 2.1 Let ${}^h e \gg 0$ and ${}^h am > 0$.

For a working consumer at age h ($h = 1, \dots, \tau$), let ${}^h x = ({}^h x_1, \dots, {}^h x_{\tau+v-h}) \in R_+^{(\tau+v-h)l}$ be a consumption stream. Let $(v/\tau) {}^h am \in R_+$ be fixed money holding for a consumer to withdraw from his income at each period till his retirement. ${}^h am$ is average income during his working periods. For the retired consumer at age h ($h = \tau + 1, \dots, \tau + v$), let ${}^h x = ({}^h x_1, \dots, {}^h x_{\tau+v+1-h}) \in R_+^{(\tau+v+1-h)l}$ be a consumption stream. At period $t = 1$, money price is one and a vector of market prices is denoted as $p_1 \in \text{int } R_+^l$, where p_1 is a vector of monetary market prices of perishable goods. At period t , a vector of market prices is denoted as $p_t \in \text{int } R_+^l$.

An action is a decision to trade in a commodity market and a money market at period 1. We denote the action by $({}^h x_1, (v/\tau) {}^h am) \in R_+^{l+1}$, where ${}^h am = \{1/(\tau+v)\} \sum_{t=1}^{\tau} E[p_t] \cdot {}^h e$. Let $(v/\tau) {}^h am$ the amount of withdrawn fund at each working period and $v {}^h am$ is paid for v retired periods. A plan is a decision to trade in a commodity market and a money market at period t . We denote the plan by $({}^h x_t, (v/\tau) {}^h am) \in R_+^{l+1}$.

Let $q = (q_2, \dots, q_v)$ a price vector for forward contracts on goods for $v - 1$ periods. Forward contracts on commodities at period 1 by a consumer at age h is ${}^h c = ({}^h c_2, \dots, {}^h c_v) \in R^{(v-1)l}$. When the consumer contracts to buy the k th good ${}^h c_{ik} > 0$ at period t in the forward market, he receives the k th good ${}^h c_{ik}$ at the beginning of period t . When he contracts to sell the k th good ${}^h c_{ik} < 0$, he delivers it. We assume that

the consumer must settle his forward contracts within his own account at period t , as follows:

Assumption 2.2 $q \cdot {}^h c = 0$.

In the spot and forward markets, given a price vector p_1 , an endowment ${}^h e_1$, and an average income ${}^h am$, the consumer chooses an action $({}^h x_1, {}^h am)$ restricted by a subset of an action space R_+^{t+1} and a contract ${}^h c$ by a subset of a contract space $R^{(v-1)t}$, which are denoted by ${}^h \beta_1(p_1) = \{({}^h x_1, {}^h am) \in R_+^{t+1} \mid p_1 \cdot {}^h x_1 + (v/\tau) {}^h am \leq p_1 \cdot {}^h e_1\}$ and ${}^h \beta_1(q) = \{{}^h c_1 \in R^{(v-1)t} \mid q \cdot {}^h c = 0\}$. A plan is restricted by each subset ${}^h \beta_t$ of the plan space R_+^{t+1} which depends on the chosen action ${}^h x_t$. This subset is represented as follows: ${}^h \beta_t(p_t) = \{{}^h x_t \in R_+^{t+1} \mid p_t \cdot {}^h x_t + (v/\tau) {}^h am \leq p_t \cdot ({}^h e_t + {}^h c_t)\}$ for $2 \leq t \leq \tau$ and ${}^h \beta_t = \{{}^h x_t \in R_+^{t+1} \mid p_t \cdot {}^h x_t \leq {}^h d_t\}$ for $\tau+1 \leq t \leq \tau+v$.

On the other hand, a retired consumer's action is defined as a decision of trade in markets at period 1 and represented by ${}^h x_1 \in R_+^1$. Given a price vector p_1 , the consumer chooses an action restricted by a subset of an action space R_+^1 , which is denoted by ${}^h \beta_1(p_1) = \{{}^h x_1 \in R_+^1 \mid p_1 \cdot {}^h x_1 \leq {}^h d_1\}$, where ${}^h d_1$ is a pension. For $\tau+2 \leq t \leq \tau+v$, there is a plan which is restricted by each subset ${}^h \beta_t(p_t) = \{{}^h x_t \in R_+^{t+1} \mid p_t \cdot {}^h x_t \leq {}^h d_t\}$ of the plan space R_+^{t+1} .

We assume that consequences of his consumption behavior at period 1 are under certainty and ones at period t ($t = 2, \dots, \tau+v$) are under uncertainty. We adopt the expected utility hypothesis with respect to the consumer's preferences to the consumption stream.

A space of consequences of the consumer's action and plan at age h is the space of consumption stream $C = R_+^{(\tau+v-h)t}$. Let $M(R_+^t)$ the space of all probability measures on the measures on the measurable space $(R_+^t, B(R_+^t))$, where $B(R_+^t)$ is the Borel σ -field on R_+^t . On the Cartesian product $M^{\tau+v-h}(R_+^t)$ of the spaces $M(R_+^t)$, the consumer chooses a probability measure and the preferences are represented by a preordering $\succeq^t$. Thus we assume as follows [7, Assumption 1, 6]:

Assumption 2.3 There exists a von Neumann-Morgenstern utility ${}^h u : C \rightarrow R$ which is continuous, bounded, such that the mapping ${}^h v : M^{\tau+v-h}(R_+^t) \rightarrow R$ defined by ${}^h v(\mu) = \int_C {}^h u d\mu$ for any $\mu \in M^{\tau+v-h}(R_+^t)$ is a representation of the preferences $\succeq^t$. Any von Neumann-Morgenstern utility ${}^h u$ is concave and monotone.

Under Assumption 2.2, the consumer at age h is faced on the certainty at period 1 and the uncertainty at t . Then, according to [14], a mapping ${}^h v(\mu)$ has the following form for any $\mu = (\mu_1, \dots, \mu_{\tau+v-h}) \in M^{\tau+v-h}(R_+^t)$.

Assumption 2.4 ${}^h v(\mu) = {}^h u_1 + \sum_{t=2}^{\tau+v-h} \int_{R_+^t} {}^h u_t d\mu_t$.

The uncertainty in Assumptions 2.3 and 2.4 means one of consequences when the consumer

will act in the future period. On the other hand, the consumer expects the next period's equilibrium price system. According to Grandmont's exogenous expectation [7, Assumption 2], we assume a Grandmont's expectation function that the present price system quoted $p^1 = (p_1, 1, q) \in R_+^{l+1+(v-1)l} \setminus \{0\}$ corresponds to a probability distribution ${}^h\psi_t(p^1) \in M(R_+^l \setminus \{0\})$.

Assumption 2.5 Let the expectation function ${}^h\psi_t(p^1) : R_+^{l+1+(v-1)l} \setminus \{0\} \rightarrow M(R_+^l \setminus \{0\})$, $t=2, \dots, v$.

- (1) $\psi_t(p^1)$ is continuous, when $M(R_+^l \setminus \{0\})$ is endowed with the topology of weak convergence of probability measures.
- (2) For any $p^1 \in R_+^{l+1+(v-1)l} \setminus \{0\}$, $\text{int co supp } {}^h\psi_t(p^1) \neq \emptyset$.
- (3) For any $p^1 \in R_+^{l+1+(v-1)l} \setminus \{0\}$, ${}^h\psi_t(p^1)(\text{int } R_+^l \setminus \{0\}) = 1$.

Since we introduce money to the model, according to Grandmont [5, (c. 3), p. 222] we assume a further assumption.

Assumption 2.6 For any $p^1 \in R_+^{l+1+(v-1)l} \setminus \{0\}$, the set $\{ {}^h\psi_t(p^1) \mid p^1 \in R_+^{l+1+(v-1)l} \setminus \{0\} \}$, $t = 2, \dots, v$ is relatively (weakly) compact.

This assumption is the necessary condition for the boundary condition and at least one consumer shall satisfy this for the proof of the existence of equilibrium in the spot market. All consumers assume assumptions 2.3 and 2.4 for the existence of equilibrium in the forward market.

Well, we shall show that two uncertainties of consequences of consumer's actions at the future period and expectation of spot price system at the future period are related by the assumption of a random variable γ . We assume that the consumer at age h has the following continuous linear mapping γ , which we call a transformation function.

Assumption 2.7 $\gamma : {}^h a_1 \cdot \prod_{t=1}^{\tau+v-h} {}^h a_t (R_+^{l+1} \setminus \{0\}) \rightarrow C$, where ${}^h a_t (R_+^{l+1} \setminus \{0\})$ denotes the range of a measurable function ${}^h a_t : R_+^{l+1} \setminus \{0\} \rightarrow R_+^l$.

Since the transformation function γ is a random variable because a price vector p_t is a random variable by Assumption 2.3, a probability distribution defined on the space of consequences C from the space of price expectation $R_+^{l+1+(v-1)l} \setminus \{0\}$ can be induced by the random variable γ . According to [13], a von Neumann-Morgenstern utility function is denoted as ${}^h u(\gamma({}^h a_1 \cdot \prod_{t=1}^{\tau+v-h} {}^h a_t)) = \sum_{t=1}^{\tau+v-h} {}^h u_t({}^h a_t(p_t))$ by the function γ . Therefore, an expected utility function on ${}^h a_1$ and p^1 , ${}^h v({}^h a_1, p^1)$ is denoted by

$${}^h v({}^h a_1, p^1) = {}^h u_1({}^h x_1) + \sum_{t=2}^{\tau+v-h} \int_{R_+^l} {}^h u_t({}^h a_t(p_t)) d {}^h \psi_t(p^1). \quad 2.1$$

By the relation 2.1, a rule of Grandmont's decision making [7, p. 884] is formulated as follows:

Given p^1 , for any actions ${}^h a_1, {}^h a_1' \in {}^h \beta_1(p_1)$, ${}^h a_1 \succeq_{p^1} {}^h a_1'$ if and only if ${}^h v({}^h a_1, p^1) \geq$

$$^h u(a_1, \dots, p^1).$$

2. 2

We shall assume the condition of the solvency so that the consumer will not be bankrupt at period t . The condition of the solvency at period t is:

Assumption 2.8 $p_t \cdot (^h e_t + ^h c_t) \geq 0$, for any $p_t \in \text{supp } ^h \psi_t(p^1)$.

3. Generations' Optimality in Goods Spot and Forward Markets

Each consumer decides on an action and a plan. We shall show each generation's optimality and each demand correspondence's properties.

The consumer decides on an action $(^h x_1, (v/\tau) ^h am)$ and a contract $^h c$ at period 1 and a plan $^h x_t$ at period t subject to the sets of his budgets $^h \beta_1(p_1)$, $^h \beta_1(q)$, $^h \beta_t(p_t)$, respectively. In the first step, we solve a problem to decide on the action $^h x_1$ and the plan $^h x_t$, given a price vector p_t , an endowment $^h e_1$ for $t = 1$, an endowment $(^h e_t + ^h c_t)$ for $2 \leq t \leq v$, $^h e_t$ for $v + 1 \leq t \leq \tau$, $^h d_t$ for $\tau + 1 \leq t \leq \tau + v$ and an average income $^h am$ subject to the set of budget $^h \beta_t(p_t)$. We substitute the solution $^h x_t^*$ for the equation 2. 1 and we shall obtain an expected utility function $^h v(^h a_1, p_1)$. In the second step, we shall obtain the optimal action $^h a_1^* = (^h x_1^*, (v/\tau) ^h am, ^h c^*)$ subject to the sets of budget $^h \beta_1(p_1)$ and $^h \beta_1(q)$ in the goods spot market and goods forward market.

3. 1 The working generations' optimality

The working consumer decides on an action $^h a_1$ and a plan $^h a_t$ subject to the sets of his budgets $^h \beta_t(p_t)$ with two steps. In the first step, we shall solve a problem to decide on the action $^h a_1$ and the plan $^h a_t$, given the price p_t , subject to the set of budget $^h \beta_t(p_t)$. When we substitute the solution a_t^* for the equation 2. 2, we shall obtain an expected utility function $^h v(^h a_1, p_1)$. In the second step, we shall solve the action a_1 subject to the set of budget $^h \beta_1(p_1)$.

First, we solve the problem of plan-decision. Then, we obtain the solution of action-problem. Next, given the price p_1 and the action a_1 at period 1, we solve a well known problem to decide on the plan a_t subject to the set of budget, $^h \beta_t(p_t)$.

Problem 3.1 Under Assumptions 2. 1 to 2. 7, $p^1 \gg 0$, at age h ($1 \leq h \leq \tau$), given the average fund money $^h am$ and $^h c$,

$$\max_{\{^h x_t\}} \sum_{t=1}^{\tau+v+1-h} ^h u_t(^h x_t),$$

subject to $p_1 \cdot ^h x_1 + (v/\tau) ^h am = p_1 \cdot ^h e_1$, $p_2 \cdot ^h x_2 + (v/\tau) ^h am = p_2 \cdot (^h e_2 + ^h c_2)$, ..., $p_v \cdot ^h x_v + (v/\tau) ^h am = p_v \cdot (^h e_v + ^h c_v)$, ..., $p_\tau \cdot ^h x_\tau + (v/\tau) ^h am = p_\tau \cdot ^h e_\tau$, $p_{\tau+1} \cdot ^h x_{\tau+1} = ^h d_\tau$, ..., $p_{\tau+v} \cdot ^h x_{\tau+v} = ^h d_{\tau+v-1}$.

Solution Let $L_1 = ^h u_1(x_1) - \lambda_1 \{ p_1 \cdot ^h x_1 + (v/\tau) ^h am - p_1 \cdot ^h e_1 \}$.

$$\frac{\partial^h u_1}{\partial^h x_1} = \lambda_1 \cdot p_1, p_1 \cdot {}^h x_1 + (v/\tau) {}^h am - p_1 \cdot {}^h e_1 = 0.$$

$$\frac{\partial^h u_1}{\partial^h x_1}$$

Since u_1 is concave, these conditions are the necessary and sufficient for the solutions. Let the solutions ${}^h x_1^*(p_1), \lambda_1^*$ in the spot market.

Next, for $t = 2, \dots, v$, let $L_t = {}^h u_t(x_t) - \lambda_t \{ p_t \cdot {}^h x_t + (v/\tau) {}^h am - p_t \cdot ({}^h e_t + {}^h c_t) \}$.

$$\frac{\partial^h u_t}{\partial^h x_t} = \lambda_t \cdot p_t, p_t \cdot {}^h x_t + (v/\tau) {}^h am - p_t \cdot ({}^h e_t + {}^h c_t) = 0.$$

$$\frac{\partial^h u_t}{\partial^h x_t}$$

Since u_t is concave, these conditions are the necessary and sufficient for the solutions. Let the solutions ${}^h x_t^*(p_t, {}^h c_t), \lambda_t^*$.

For $t = v+1, \dots, \tau$, there is no forward market for the consumer. Let $L_t = {}^h u_t(x_t) - \lambda_t \{ p_t \cdot {}^h x_t + (v/\tau) {}^h am - p_t \cdot {}^h e_t \}$.

$$\frac{\partial^h u_t}{\partial^h x_t} = \lambda_t = p_t, p_t \cdot {}^h x_t + (v/\tau) {}^h am - p_t \cdot {}^h e_t = 0.$$

$$\frac{\partial^h u_t}{\partial^h x_t}$$

Let the solutions ${}^h x_t^*(p_t), \lambda_t^*$.

Lastly, for the retirement period, let $L_t = {}^h u_t(x_t) - \lambda_t \{ p_t \cdot {}^h x_t - {}^h d_{t-1} \}$.

$$\frac{\partial^h u_t}{\partial^h x_t} = \lambda_t \cdot p_t, p_t \cdot {}^h x_t - {}^h d_{t-1} = 0.$$

$$\frac{\partial^h u_t}{\partial^h x_t}$$

Let the solutions ${}^h x_t^*(p_t), \lambda_t^*$. □

Let ${}^h x_t^*$ be a solution for Problem 3. 1. In Section 2, we have defined an expected utility function ${}^h v(a_1, p_1)$. When we substitute the solution ${}^h x_t^*$ for the equation 2. 2, we shall obtain an expected utility function ${}^h v(a_1, p_1) = {}^h u_1(x_1) + \sum_{t=1}^{\tau+v+1-h} \int {}^h u_t({}^h x_t^*) d\psi_t(p_1)$. The solution ${}^h x_1^*(p_1)$ has been decided. We decide on the contract hc as follows:

Problem 3. 2 Under Assumptions 2. 1 to 2. 8, given $q \gg 0$, at age $h(1 \leq h \leq \tau)$,

$$\max_{\{^h c\}} \sum_{t=2}^{\tau+v+1-h} \int {}^h u_t({}^h x_t^*(p_t)) d^h \psi_t(p_1), \text{ subject to } q \cdot {}^h c = 0.$$

Solution Let $L = \sum_{t=2}^{\tau+v+1-h} \int {}^h u_t({}^h x_t^*(p_t, {}^h c_t)) d^h \psi_t(p_1) - \lambda q \cdot {}^h c$.

$$\frac{\partial}{\partial^h c} \int {}^h u_t({}^h x_t^*(p_t, {}^h c_t)) d^h \psi_t(p_1) = \lambda \cdot q, q \cdot {}^h c = 0.$$

$$\frac{\partial^h L}{\partial^h c_t}$$

Since u_t is concave, these conditions are the necessary and sufficient for the solutions. Let the solutions ${}^h c^*, \lambda^*$. □

Let ${}^h v({}^h c, p_1) = \sum_{t=2}^{\tau+v+1-h} \int {}^h u_t({}^h x_t^*(p_t, {}^h c_t)) d^h \psi_t(p_1)$.

The necessary and sufficient condition for the existence of an optimal solution to the next problem 3. 2 is:

Assumption 3.3 $q_t \in \text{int co supp } {}^h\psi_t(p_1)$.

Theorem 3.1 The necessary and sufficient conditions for the existence of an optimal solution to Problem 3.2 are $p^1 \gg 0$, $q_t \in \text{int co supp } {}^h\psi_t(p^1)$.

Proof See Grandmont [6, Proposition 2] and Nishimura [17, Theorem 4.1]. $\square$

For a consumer, we assume that consumption set R^{+l} is non-empty and compact. We define a correspondence ${}^{ih}\beta_1(p_1) = \{{}^{ih}a_1 \in R_+^{l+1} \mid p_1 \cdot {}^{ih}x_1 + {}^{ih}am \leq p_1 \cdot {}^{ih}e_1\}$ from $R_+^l \times R_+$ to R_+^{l+1} and a correspondence ${}^{ih}\beta_1(p_1) = \{{}^{ih}a_t \in R_+^l \mid p_t \cdot {}^{ih}x_t \leq {}^{ih}d_t\}$ from $R_+^l \times R_+$ to R_+^l . By Debreu [3], the correspondences have the following properties:

Lemma 3.1 Correspondences ${}^{ih}\beta_1$, ${}^{ih}\beta_t$ are non-empty, convex-valued, compact-valued and continuous on R_+^{l+1} and R_+^l respectively.

Proof Debreu[3, Lemma 3]. $\square$

A rule of Grandmont's decision making is that given p^1 , for any actions ha_1 and ${}^ha_1' \in {}^h\beta_1(p_1)$, ${}^ha_1 \succeq_{p^1} {}^ha_1'$ if and only if ${}^h\nu({}^ha_1, p_1) \geq {}^h\nu({}^ha_1', p_1)$. Therefore, for i th consumer at age h , maximizing ${}^{ih}\nu({}^ha_1, p_1)$ on ${}^{ih}\beta_1(p_1)$ yields the set ${}^{ih}\xi_1(p_1) = \{{}^{ih}a_1 \in R_+^{l+1} \mid {}^{ih}\nu({}^{ih}a_1, p_1) \geq {}^{ih}\nu({}^{ih}a_1', p_1) \text{ for every } {}^{ih}a_1'\}$. We call ${}^{ih}\xi_1(p_1)$ his demand correspondence. The demand correspondence ${}^{ih}\xi_1(p^1)$ has the following properties.

Lemma 3.2 ${}^{ih}\xi_1$ is nonempty, convex-valued, compact-valued and upper hemi-continuous. For all $p_1 \in R_+^{l+1}$ and all ${}^{ih}a_1 \in {}^{ih}\xi_1$, $p_1 \cdot {}^{ih}x_1 + {}^{ih}am = p_1 \cdot {}^{ih}e_1$.

Proof By Lemma 3.1, ${}^{ih}\xi_1$ is nonempty, convex-valued, compact-valued and upper hemi-continuous. From the monotonicity of preferences, his budget satisfies the equality [19, Proposition 7.4 and Corollary]. $\square$

By Grandmont [5, Proposition 4.2], the consumer's demand correspondence ${}^{ih}\xi_1$ satisfies the following boundary condition.

Lemma 3.3 For any ${}^{ih}a_1^k \in {}^{ih}\xi_1(p_1^k)$, if p_1^k tends to $p_1^0 \in R_+^{l+1}/P_1$, or if $\|p_1^k\|$ tends to $+\infty$, then $\|{}^{ih}a_1^k\|$ tends to $+\infty$.

In the goods forward market, we show properties of ${}^{ih}\beta_1(q) = \{{}^hc \in R^{(v-1)l} \mid q \cdot {}^hc = 0, p_2 \cdot (e_2 + c_2) + m_1^* \geq 0, p_t \cdot ({}^he_t + {}^hc_t) \geq 0, \text{ for any } p_t \in \text{supp } {}^h\psi_t(p^1)\}$, according to the results by Debreu [3]. We define the i th trader's excess demand correspondence ${}^{ih}\xi_1$ and prove the properties of ${}^{ih}\xi_1$.

The correspondence ${}^{ih}\beta_1(q)$ has the following properties:

Lemma 3.4 Correspondence ${}^{ih}\beta_1$ is non-empty, convex-valued, compact-valued and continuous on $R^{l(v-1)}$.

Proof Nishimura[17, Lemma 5.1]. □

A rule of Grandmont's decision making is that given p_1 , for any actions $c, c' \in {}^{ih}\beta_1(q)$, ${}^{ih}c \succeq_{p^1} c'$ if and only if ${}^{ih}v({}^{ih}c, p^1) \geq {}^{ih}v({}^{ih}c', p^1)$. Therefore, maximizing ${}^{ih}v$ on ${}^{ih}\beta_1(q)$ yields the set ${}^{ih}\xi_1(q) = \{ {}^{ih}c \in R^{l(v-1)} \mid {}^{ih}v({}^{ih}c, p^1) \geq {}^{ih}v({}^{ih}c', p^1) \text{ for every } {}^{ih}c' \}$. We call ${}^{ih}\xi_1(q)$ his demand correspondence. The demand correspondence ${}^{ih}\xi_1(q)$ has the following properties.

Lemma 3.5 ${}^{ih}\xi_1$ is nonempty, convex-valued, compact-valued and upper hemicontinuous. For all $q \in R^{l(v-1)}$ and all $c \in {}^{ih}\xi_1$, $q \cdot {}^{ih}c = 0$.

Proof Nishimura[17, Theorem 3.6]. □

The trader's demand correspondence ξ_1 satisfies a boundary condition.

Lemma 3.6 For any ${}^{ih}c^k \in {}^{ih}\xi_1(q^k)$, if q^k tends to $q^0 \in R^{l(v-1)} \setminus \text{int } R^{l(v-1)}$, or if $\|q^k\|$ tends to $+\infty$, then $q^k \cdot {}^{ih}c^k$ tends to $+\infty$.

Proof Grandmont [5, Proposition 4.2] □

3.2 The retired generation's optimality

Since the retired consumer does not face on the uncertainty at period t , his problem to obtain an optimal consumption at period 1 is well known. We assume that u_1 is continuous, concave, bounded real-valued utility function.

Problem 3.3 Under Assumptions 2.1 to 2.7, $p^1 \gg 0$, at age $h(\tau+1 \leq h \leq \tau+v)$,
 $\max_{\{x_t\}} \sum_{t=1}^{\tau+v+1-h} {}^h u_t({}^h x_t)$, subject to $p_1 \cdot {}^h x_1 = {}^h d_{h-1}, \dots, p_{\tau+v+1-h} \cdot {}^h x_{\tau+v+1-h} = {}^h d_{h-1}$.

Solution Let $L = {}^h u_t({}^h x_t) - \lambda \{p_t \cdot {}^h x_t - {}^h d_{h-1}\}$.

$$\frac{\partial {}^h u_t}{\partial {}^h x_t} = \lambda \cdot p_t, p_t \cdot {}^h x_t - {}^h d_{h-1} = 0.$$

$$\frac{\partial {}^h x_t}{\partial {}^h x_t}$$

Since ${}^h u_t$ is concave, these conditions are the necessary and sufficient for the solutions. Let the solutions ${}^h x_t^*(p_t), \lambda^*$. □

Problem 3.4 Under Assumptions 2.1 to 2.8, $p^1 \gg 0$, at age $h(\tau+1 \leq h \leq \tau+v)$,
 $\max_{\{x_t^*\}} \sum_{t=1}^{\tau+v+1-h} \int {}^h u_t({}^h x_t^*(p_t)) d {}^h \psi_t(p^1)$

Solution Since ${}^h u_t$ is concave, Jensen's inequality,

$$\sum_{t=1}^{\tau+v+1-h} \int u_t(hx_t^*(p_t)) d^h \psi_t(p^1) \leq \sum_{t=1}^{\tau+v+1-h} u_t(E[hx_t^*(p_t)]).$$

Let the solutions $^h x_t^{**} = E[hx_t^*(p_t)]$. □

4. Definitions and Assumptions in Bonds Spot and Forward Markets

4.1 The Consumers' actions and plans in Bonds Spot and Forward Markets

We consider an exchange economy with a discount bond of one period and a government bond of v periods issued newly in bond spot markets and there is government bond forward markets for $v-1$ periods. We define actions and plans formed by consumers in a discount bond and government bonds spot and forward markets.

A discount bond has a maturity term 1 and pays one money unit per one unit of bond. A government bond has a maturity term v and pays an fixed interest $C^0 = (C_1^0, \dots, C_{v-1}^0)$ money unit per one unit of bond at each period till the maturity of v periods and redeem one money unit per one unit of bond.

At each period there exist a discount bond and government bonds. A consumer who works for τ periods to withdraw a fixed average fund $(v/\tau) am > 0$ which is perfectly foresighted. A consumer at age $h (1 \leq h \leq \tau)$ who works for τ periods is endowed with $^h b^1 \in R_+^{h-1}$ which is perfectly foresighted, where R_+^{h-1} is a closed positive cone of R^{h-1} . At period $t = 1$, every consumer has no debt. The retired are endowed with their fund $^h d_{h-1}$, which has been accumulated for the working period. At period $t = 1$, every consumer has no debt. For simplicity, all the consumers cannot borrow and lend money from the others. We assume

Assumption 4.1 Let $^h b^1 > 0, (v/\tau) am > 0$ at age $h (1 \leq h \leq \tau)$. Let $^h d_t > 0$ at age $h (\tau+1 \leq h \leq \tau+v)$ and at period $t (1 \leq t \leq v-1)$.

A consumer at age 1 holds the bond term structure as follows:

age	period	The left side of budget	The right side of budget
$h = 1,$	$t = 1$	$^1 b^1 = (^1 b_v^1)$	$^1 b^0 = 0$
	$t = 2$	$^1 b^2 = (^1 b_{v-1}^1, ^1 b_v^2)$	$^1 b^1 = (^1 b_{v-1}^1)$
	$t = 3$	$^1 b^3 = (^1 b_{v-2}^1, ^1 b_{v-1}^2, ^1 b_v^3)$	$^1 b^2 = (^1 b_{v-2}^1, ^1 b_{v-1}^2)$
		. . .	
	$t = v$	$^1 b^v = (^1 b_1^1, \dots, ^1 b_v^v)$	$^1 b^v = (^1 b_1^1, \dots, ^1 b_{v-1}^{v-1})$
	$t = v+1$	$^1 b^{v-1} = (^1 b_1^1, \dots, ^1 b_v^v)$	
		. . .	
	$t = \tau$	$^1 b^\tau = (^1 b_1^{\tau-1-v}, \dots, ^1 b_v^{\tau-1}), ^1 d_\tau$	
	$t = \tau+1$	$^1 b^{\tau+1} = (^1 b_1^{\tau+1-v}, \dots, ^1 b_{v-1}^\tau), ^1 d_{\tau+1}$	
	$t = \tau+2$	$^1 b^{\tau+1} = (^1 b_1^{\tau+2-v}, \dots, ^1 b_{v-2}^{\tau+1}), ^1 d_{\tau+2}$	
		. . .	

$$\begin{aligned} t = \tau + v - 1 \quad {}^1b^{\tau+1-1} &= {}^1b_1^{\tau+v-1}, {}^1d_{\tau+v-1} \\ t = \tau + v \quad {}^1b^{\tau+1} &= 0 \end{aligned}$$

At period $t = 1$, a vector of market price of discount bond is denoted as $p_{dt} \in \text{int } R_+$. At age h ($\tau \leq h \leq \tau + v - 1$), a consumer purchases a discount bond for the next period payment by the redemption of maturity and coupons of bond. Let a discount rate of interest r_t , ($1 \leq t \leq v$). Since the discount price $p_{dt} > 0$, the rate r_t is $r_t \in (-1, \infty)$.

p_{dt} is denoted as follows:

$$p_{dt} = \frac{1}{1 + r_t}.$$

At period $t = 1$, a new government bond pays a coupon C_v^1 per one unit of bond and redeem at a maturity of period v . At period $t = 1$, a vector of market prices is denoted as $p_b^1 = (p_{b1}^1, \dots, p_{bv}^1) \in \text{int } R_+^{v1}$. p_b^1 is defined as follows:

$$\begin{aligned} p_{b1}^1 &= \frac{C_1^0 + 1}{1 + r_1}, \\ p_{b2}^1 &= \frac{C_2^0}{1 + r_1} + \frac{C_2^0 + 1}{(1 + r_1)(1 + r_2)}, \\ &\dots, \\ p_{bv}^1 &= \frac{C_{v-1}^0}{1 + r_1} + \frac{C_{v-1}^0}{(1 + r_1)(1 + r_2)} + \dots + \frac{C_{v-1}^0 + 1}{(1 + r_1) \cdots (1 + r_{v-1})}, \\ p_{bv}^1 &= \frac{C_v^1}{1 + r_1} + \frac{C_v^1}{(1 + r_1)(1 + r_2)} + \dots + \frac{C_v^1 + 1}{(1 + r_1) \cdots (1 + r_v)}. \end{aligned}$$

An action is a decision to trade in bond spot and forward markets at period 1. We denote the action by ${}^h a_1 = {}^h b^{v-h} \in R_+^{v-h}$ for $1 \leq h \leq v$ and ${}^h a_1 = {}^h b^{\tau-h} \in R_+^{\tau-h}$ for $v+1 \leq h \leq \tau$ in bond spot market and in bond forward for $2 \leq h \leq \tau$. A plan is a decision to trade in bond spot and forward markets for $2 \leq t \leq \tau + \tau$. We denote the plan by ${}^h a_t = {}^h b_t \in R_+^t$ for $2 \leq h \leq \tau$ and by ${}^h a_t = ({}^h d^t, {}^h b^t) \in R_+^{\tau+v-t}$ for $v+1 \leq h \leq \tau + v$.

Let $q_b = (q_{b2}, \dots, q_{bv})$ a price vector for forward contracts on bonds for $v-1$ periods. Forward contracts on bonds at period 1 by a consumer at age h is ${}^h c_b = ({}^h c_{b2}, \dots, {}^h c_{bv}) \in R^{(v-1)1}$. When the consumer contracts to buy the k th bond ${}^h c_{tk} > 0$ at period t in the forward market, he receives the k th bond ${}^h c_{tk}$ at the beginning of period t . When he contracts to sell the k th bond ${}^h c_{tk} < 0$, he delivers it. We assume that the consumer must settle his forward contracts within his own account at period t , as follows:

Assumption 4.2 $q_b \cdot {}^h c_b = 0$.

In the spot and forward markets, given a price vector p^1 and an endowment ${}^h b^1$, (v/τ) ${}^h a_m$ and ${}^h d_{h-1}$, the working consumer at age h ($v+1 \leq h \leq \tau$) chooses an action ${}^h b^1$ and a contract ${}^h c_b$ restricted by subsets of an action space R_+^v and a contract space $R^{(v-1)1}$, which are denoted by ${}^h \beta_1(p_{b1}) = \{{}^h b^1 \in R_+^v \mid$

$p_{b1} \cdot {}^h b^1 \leq p_{b1} \cdot {}^h b^0 + (v/\tau) {}^h a_m + C^0 \cdot {}^h b^0$, where $C^0 = (C_1^0, \dots, C_{v-1}^0)$ and ${}^h \beta_1(q_b) = \{{}^h c_b \in R^{(v-1)} \mid q_b \cdot {}^h c_b = 0\}$. A plan is restricted by each subset ${}^h \beta_t$ of the plan space R_+^v which depends on the chosen action ${}^h b^t$. This subset is represented as follows: ${}^h \beta_t(p_{bt}) = \{{}^h b^t \in R_+^v \mid p_{bt} \cdot {}^h b^t \leq p_{bt} \cdot ({}^h b^{t-1} + {}^h c_t) + (v/\tau) {}^h a_m + {}^h b_0^{t-1} + C^0 \cdot {}^h b^{t-1}\}$ for $2 \leq t \leq \tau - 1$, ${}^h \beta_\tau(p_{b\tau}) = \{{}^h b^\tau \in R_+^v \mid p_{d\tau} \cdot {}^h d_\tau + p_{b\tau} \cdot {}^h b^\tau \leq p_{b\tau} \cdot ({}^h b^{\tau-1} + {}^h c_\tau) + (v/\tau) {}^h a_m + {}^h b_0^{\tau-1} + C^0 \cdot {}^h b^{\tau-1}\}$ and ${}^h \beta_{\tau+1} = \{{}^h d_t \in R_+ \mid p_{d1} \cdot {}^h d_t \leq {}^h b_0^{t-1} + C_0 \cdot {}^h b_t - 1\}$ for $\tau + 1 \leq t \leq \tau + v$.

On the other hand, a retired consumer's action for $\tau + 1 \leq h \leq \tau + v$ is defined as a decision of trade in discount bond spot market at period 1 and represented by ${}^h d_1 \in R_+$. Given a price vector p_{d1} , the consumer chooses an action restricted by a subset of an action space R_+ , which is denoted by ${}^h \beta_1(p_{d1}) = \{{}^h d_1 \in R_+ \mid p_{d1} \cdot {}^h d_1 \leq {}^h b_0^0 + C^0 \cdot {}^h b^0\}$ and ${}^h \beta_1(q_b) = \{{}^h c_b \in R^{(v-1)} \mid q_b \cdot {}^h c_b = 0\}$. There is a plan which is restricted by each subset ${}^h \beta_t(p_{dt}) = \{{}^h d_t \in R_+ \mid p_{dt} \cdot {}^h d_t + p_{bt} \cdot {}^h b^t \leq p_{bt} \cdot ({}^h b^{t-1} + {}^h c_t) + {}^h b_0^{t-1} + C^0 \cdot {}^h b^{t-1}\}$ for $2 \leq t \leq v - 2$ and ${}^h \beta_{v-1}(p_{dv-1}) = \{{}^h d_{v-1} \in R_+ \mid p_{dv-1} \cdot {}^h d_{v-1} \leq {}^h b_0^{v-2} + C^0 \cdot {}^h b^{v-2}\}$ for $t = v - 1$.

We assume that consequences of his consumption behavior at period 1 are under certainty and ones at period t ($2 \leq t$) are under uncertainty. We adopt the expected utility hypothesis with respect to the consumer's preferences to the bond stream.

A space of consequences of the consumer's action and plan is the space of bond stream $B = R_+^{v+1}$.

Let $M(R_+^{v+1})$ be the space of all probability measures on the measures on the measurable space $(R_+^{v+1}, B(R_+^{v+1}))$, where $B(R_+^{v+1})$ is the Borel σ -field on R_+^{v+1} . On the Cartesian product $M^h(R_+^{v+1})$ of the spaces $M(R_+^{v+1})$, the consumer chooses a probability measure and the preferences are represented by a preordering $\succeq^l$. Thus we assume as follows [6, Assumption 1, 6]:

Assumption 4.3 There exists a von Neumann-Morgenstern utility ${}^h u : C \rightarrow R$ which is continuous, bounded, such that the mapping ${}^h v : M^{\tau+v-h}(R_+^{v+1}) \rightarrow R$ defined by ${}^h v(\mu) = \int_C {}^h u \, d\mu$ for any $\mu \in M^{\tau+v-h}(R_+^{v+1})$ is a representation of the preferences $\succeq^l$. Any von Neumann-Morgenstern utility ${}^h u$ is concave and monotone.

Under Assumption 4.3, the consumer is faced on the certainty at period 1 and the uncertainty at t . Then, according to [13], a mapping ${}^h v(\mu)$ has the following form for any $\mu = (\mu_1, \dots, \mu_{\tau+v-h}) \in M^{\tau+v-h}(R_+^{v+1})$.

Assumption 4.4 ${}^h v(\mu) = {}^h u_1 + \sum_{t=2}^{\tau+v-h} \int_{R_+^{v+1}} {}^h u_t \, d\mu_t$.

According to Grandmont's exogenous expectation [6, Assumption 2], we assume a Grandmont expectation function that the present price system quoted $(p_d^1, p_b^1, q_b) \in R_+^{1+v+(v-1)} \setminus \{0\}$ is correspond to a probability distribution ${}^h \psi_t(p_d^1, p_b^1, q_b) \in M(R_+ \setminus \{0\})$.

Assumption 4.5 Let the expectation function ${}^h \psi_t(p_d^1, p_b^1, q_b) : R_+^{1+v+(v-1)} \setminus \{0\} \rightarrow M(R_+ \setminus \{0\})$, $t=2, \dots, v$.

- (1) $\psi_t(p_d^1, p_b^1, q_b)$ is continuous, when $M(R_+^{v+1} \setminus \{0\})$ is endowed with the topology of weak convergence of probability measures.
- (2) For any $(p_d^1, p_b^1, q_b) \in R_+^{l+v+(v-1)} \setminus \{0\}$, $\int \text{co supp } {}^h\psi_t(p_d^1, p_b^1, q_b) \neq 0$.
- (3) For any $(p_d^1, p_b^1, q_b) \in R_+^{l+v+(v-1)} \setminus \{0\}$, $\int {}^h\psi_t(p_d^1, p_b^1, q_b) (\int R_+^{v+1} \setminus \{0\}) = 1$.

Since we introduce money to the model, according to Grandmont [5, (c. 3), p. 222] we assume a further assumption.

Assumption 4.6 For any $(p_d^1, p_b^1, q_b) \in R_+^{l+v+(v-1)} \setminus \{0\}$, the set $\text{supp } {}^h\psi_t(p_d^1, p_b^1, q_b)$, $t=2, \dots, v$ is relatively (weakly) compact.

We assume that the consumer at age h has the following continuous linear mapping γ , which we call a transformation function.

Assumption 4.7 $\gamma: {}^h a_1 \cdot \prod_{t=1}^{\tau+v-h} {}^h a_t (R_+^{v+1} \setminus \{0\}) \rightarrow B$, where ${}^h a_t (R_+^{v+1} \setminus \{0\})$ denotes the range of a measurable function ${}^h a_t: R_+^{v+1} \setminus \{0\} \rightarrow A_t$.

Since the transformation function γ is a random variable because a price vector p_t is a random variable by Assumption 4.3, a probability distribution defined on the space of consequences C from the space of price expectation $R_+^{l+v+(v-1)} \setminus \{0\}$ can be induced by the random variable γ . According to [13], a von Neumann-Morgenstern utility function is denoted as ${}^h u(\gamma({}^h a_1 \cdot \prod_{t=1}^{\tau+v-h} {}^h a_t)) = \sum_{t=1}^{\tau+v-h} {}^h u_t({}^h a_t(p_d^1, p_b^1))$ by the function γ . Therefore, an expected utility function on a^1 and p^1 , ${}^h v(a_1, ((p_d^1, p_b^1, q_b)))$ is denoted by

$${}^h v(a_1, ((p_d^1, p_b^1, q_b))) = {}^h u_1({}^h x_1) + \sum_{t=2}^{\tau+v-h} \int_{R_+^{v+1}} {}^h u_t({}^h a_t(p_t)) d {}^h \psi_t(p_d^1, p_b^1, q_b). \quad 4.1$$

By the relation 4.1, a rule of Grandmont's decision making [7, p. 884] is formulated as follows:

Given p^1 , for any actions ${}^h a_{11}, {}^h a_{11}' \in \beta_{b1}(p_1)$, ${}^h a_1 \succeq_{p^1} {}^h a_1'$ if and only if

$${}^h v({}^h a_1, (p_d^1, p_b^1, q_b)) \geq {}^h v({}^h a_1', (p_d^1, p_b^1, q_b)). \quad 4.2$$

We shall assume the condition of the solvency so that the consumer will not be bankrupt at period t . The condition of the solvency at period t is:

Assumption 4.8 $p_t \cdot ({}^h e_t + {}^h c_t) \geq 0$, for any $p_t \in \text{supp } {}^h \psi_t(p_d^1, p_b^1, q_b)$.

4.2 The Central Bank's action

The central bank has the issue right of money. The central bank settles one yen per one money unit. The government bond is issued with a maturity of v periods and fixed coupon C_v^1 per one unit at t

= 1.

The central bank at $t = 1$ pays the amount of coupons ${}^cC_0 = \sum_{i=1}^I \sum_{h=1}^{\tau+v} \sum_{t=0}^{v-1} {}^{ih}C_t^0$ and redeems the discount bond ${}^cD_0 = \sum_{i=1}^I \sum_{h=\tau+1}^{\tau+v} {}^{ih}d_0$ and the government bond of maturity ${}^cB_0 = \sum_{i=1}^I \sum_{h=1}^{\tau+v} {}^{ih}b_0^0$.

The redemption of government bond of maturity and the repayment of coupons increase the money supply. The central bank refinances the discount bond ${}^cD_1 = p_d^1 {}^cD_0$ for the repayment of coupons and the bond for the government bond ${}^cB_v = p_{bv}^1 {}^cB_0$. The central bank decides on the new coupon C_0^1 per one unit according to the real rate of interest.

The central bank's action is ${}^c a_1 = ({}^c D_1, {}^c B_v)$ and must satisfy the following accounting identity for each (p_d^1, p_{bv}^1) ,

Assumption 4.9 $(p_d^1, p_{bv}^1) \cdot (\sum_{i=1}^I \sum_{h=\tau+1}^{\tau+v} {}^{ih}d_1, \sum_{i=1}^I \sum_{h=1}^{\tau+v} {}^{ih}b_v^1) \equiv {}^c D_1 + {}^c B_v$.

The budget set of the central bank is ${}^c \beta_1(p_d^1, p_{bv}^1) = \{ {}^c a_1 \in R_+^2 \mid (p_d^1, p_{bv}^1) \cdot (\sum_{i=1}^I \sum_{h=\tau+1}^{\tau+v} {}^{ih}d_1, \sum_{i=1}^I \sum_{h=1}^{\tau+v} {}^{ih}b_v^1) - {}^c D_1 - {}^c B_v = 0 \}$. The excess supply correspondence is $\eta(p_d^1, p_{bv}^1) = (p_d^1, p_{bv}^1) \cdot (\sum_{i=1}^I \sum_{h=\tau+1}^{\tau+v} {}^{ih}d_1, \sum_{i=1}^I \sum_{h=1}^{\tau+v} {}^{ih}b_v^1) - {}^c D_1 - {}^c B_v$.

5. Generations' Optimality in Bonds Spot and Forward Markets

Each trader decides on an action and a plan. Since there exist two generations in our model, we shall show each generation's optimality and each demand correspondence's properties.

5.1 The working generations' optimality

The working consumer decides on an action a_1 and a plan a_t at age $h(1 \leq h \leq \tau)$ and at period t subject to the sets of his budgets ${}^h \beta_1(p_{b1})$, ${}^h \beta_t(p_{bt})$ with two steps. In the first step, we shall solve a problem to decide on the plan a_t , given the bond price p_{b1} and the action a_1 , subject to the set of budget ${}^h \beta_t(p_{bt})$. When we substitute the solution a_t^* for the equation 2. 1, we shall obtain a conditional expected utility function ${}^h v(a_1, p_{b1})$. In the second step, we shall solve the action a_1 subject to the set of budget ${}^h \beta_1(p_{b1})$.

First, we show properties of ${}^h \beta_1$, ${}^h \beta_t$ according to the results by Debreu [3]. Next, we solve the problem of plan-decision and prove the properties of the conditional expected utility function ${}^h v(a_1, p_{b1})$. Then, we obtain the solution of action-problem. Next, given the price p_1 and the action a_{11} at period 1, we solve a well-known problem to decide on the plan a_t subject to the set of budget ${}^h \beta_t(p_{bt})$.

Problem 5.1 Under Assumptions 4. 1 to 4. 7, $p^1 \gg 0$, at age $h(1 \leq h \leq \tau)$, given ${}^h c_b$,

$$\max_{\{ {}^h b_t \}} \sum_{t=1}^{\tau+v+1-h} {}^h u_t({}^h b_t),$$

$$\text{subject to } p_{b1} \cdot {}^h b^1 = p_{b1} \cdot {}^h b^0 + (v/\tau) {}^h am + C^0 \cdot {}^h b^0, \quad p_{b2} \cdot {}^h b^2 = p_{b2} \cdot ({}^h b^1 + {}^h c_{b2}) +$$

$$(v/\tau) {}^h am + C^0 \cdot {}^h b^1, \dots, p_{b\tau} \cdot {}^h d_\tau + p_{b\tau} \cdot {}^h b^\tau = p_{b\tau} \cdot ({}^h b^{\tau-1} + {}^h c_b^\tau) + (v/\tau) {}^h am + {}^h b_0^{\tau-1} + C^0 \cdot {}^h b^{\tau-1}, \\ p_{b\tau+1} \cdot {}^h d_{\tau+1} = {}^h b^\tau + C^0 \cdot {}^h b^\tau, \dots, p_{b\tau+v} \cdot {}^h d = {}^h b^{\tau+v-1}.$$

Solution In the first step, let $L_1 = {}^h u_1({}^h b^1) - \lambda_1 \{ p_{b1} \cdot {}^h b_1 - p_{b1} \cdot {}^h b^0 - (v/\tau) {}^h am - C^0 \cdot {}^h b^0 \}$.

$$\frac{\partial {}^h u_1}{\partial {}^h b_1} = \lambda_1 \cdot p_{b1}, \quad p_{b1} \cdot {}^h b^1 - p_{b1} \cdot {}^h b^0 - (v/\tau) {}^h am - C^0 \cdot {}^h b^0 = 0.$$

$$\frac{\partial {}^h b_1}{\partial {}^h b_1}$$

Since u_1 is concave, these conditions are the necessary and sufficient for the solutions. Let the solutions ${}^h x_1^*(p_1), \lambda_1^*$ in the spot market.

Next, for $t=2, \dots, v$, let $L_t = {}^h u_t({}^h b^t) - \lambda_t \{ p_{bt} \cdot {}^h b^t - p_{bt} \cdot ({}^h b^{t-1} + {}^h c_t) - (v/\tau) {}^h am - {}^h b_0^{t-1} - C^0 \cdot {}^h b^{t-1} \}$.

$$\frac{\partial {}^h u_t}{\partial {}^h b^t} = \lambda_t \cdot p_{bt},$$

$$\frac{\partial {}^h b^t}{\partial {}^h b^t}$$

$$p_{bt} \cdot {}^h b^t - p_{bt} \cdot ({}^h b^{t-1} + {}^h c_t) - (v/\tau) {}^h am - {}^h b_0^{t-1} - C^0 \cdot {}^h b^{t-1} = 0.$$

Since ${}^h u_t$ is concave, these conditions are the necessary and sufficient for the solutions. Let the solutions ${}^h b_t^*(p_{bt}, {}^h c_t), \lambda_t^*$.

Last, for the retirement period, let $L_t = {}^h u_t(x_t) - \lambda_t \{ p_{bt} \cdot {}^h d_t - {}^h b^{t-1} - C^0 \cdot {}^h b^{t-1} \}$.

$$\frac{\partial {}^h u_t}{\partial {}^h x_t} = \lambda_t \cdot p_{dt}, \quad p_{dt} \cdot {}^h d_t - {}^h b^{t-1} - C^0 \cdot {}^h b^{t-1} = 0.$$

$$\frac{\partial {}^h x_t}{\partial {}^h x_t}$$

Let the solutions ${}^h d_t^*(p_{dt}), \lambda_t^*$. □

Next, we shall solve the action a_1 , subject to the set of budget $\beta_1(p_1)$.

Problem 5.2 Under Assumptions 4. 1 to 4. 7, given $q_b \gg 0$, at age $h(1 \leq h \leq \tau)$,

$$\max \sum_{t=2}^{\tau+v+1-h} \int {}^h u_t({}^h b_t^*(p_b, c_{bt})) d {}^h \psi_t(p_d^1, p_d^1, q_b), \text{ subject to } q_b \cdot {}^h c = 0.$$

$$\{ {}^h c_b \}$$

Solution Let $L = \int {}^h u_t^*({}^h b_t^*(p_{bt}, {}^h c_{bt})) d {}^h \psi_t(p_d^1, p_d^1, q_b) - \lambda q_t \cdot {}^h c_{bt}$.

$$\frac{\partial \int {}^h u_t^*({}^h b_t^*(p_{bt}, c_{bt}, m_c)) d {}^h \psi_t(p_d^1, p_d^1, q_b)}{\partial c_{bt}} = \lambda \cdot q_{bt}, \quad q_{bt} \cdot c_{bt} = 0.$$

$$\frac{\partial c_{bt}}{\partial c_{bt}}$$

Since u_t is concave, these conditions are the necessary and sufficient for the solutions. Let the solutions c_{bt}^*, λ^* . □

In the bonds spot markets, for a consumer, we assume that bond set R_+^v is non-empty and compact. We define a correspondence ${}^{ih}\beta_1(p_b^1) = \{ {}^{ih}a_{b1} \in R_+^v \mid p_{b1} \cdot {}^{ih}b^1 \leq p_{b1} \cdot {}^{ih}b^0 + (v/\tau) {}^{ih}am + C^0 \cdot {}^{ih}b^0 \}$ from R_+^v to R_+^v . By Debreu [3], the correspondence has the following properties:

Lemma 5.1 Correspondence ${}^{ih}\beta_1$ is non-empty, convex-valued, compact-valued and continuous on R_+^v .

Proof Debreu[3, Lemma 3]. □

A rule of Grandmont's decision making is that given p^1 , for any actions a_1 and $a_1' \in {}^h\beta_1(p_b^1)$, ${}^h a_{b1} \succeq {}^h a_{b1}'$ if and only if ${}^h v({}^h a_{b1}, p^1) \geq {}^h v({}^h a_{b1}', p^1)$. Therefore, for i th consumer at age h , maximizing ${}^i v$ on ${}^i\beta_1(p_{b1})$ yields the set ${}^i\zeta_1(p_{b1}) = \{{}^i a_{b1} \in R_+^v \mid {}^i v({}^i a_{b1}, p_{b1}) \geq {}^i v({}^i a_{b1}', p_{b1}) \text{ for every } {}^i a_{b1}' \}$. We call ${}^i\zeta_1(p_b^1)$ his demand correspondence. The demand correspondence ${}^i\zeta_1(p_b^1)$ has the following properties.

Lemma 5.2 ${}^i\zeta_1$ is nonempty, convex-valued, compact-valued and upper hemi-continuous. For all $p_b^1 \in R_+^v$ and all ${}^i b_1 \in {}^i\zeta_1$, $p_{b1} \cdot {}^i b^1 = p_{b1} \cdot {}^i b^0 + (v/\tau) {}^i am + C^0 \cdot {}^i b^0$.

Proof By Lemma 5. 1, ${}^i\zeta_1$ is nonempty, convex-valued, compact-valued and upper hemi-continuous. From the monotonicity of preferences, his budget satisfies the equality [19, Proposition 7. 4 and Corollary]. $\square$

By Grandmont [5, Proposition 4. 2], the consumer's demand correspondence ${}^i\zeta_1$ satisfies a boundary condition.

Lemma 5.3 For any ${}^i b^{1k} \in {}^i\zeta_1(p_b^k)$, if p_b^k tends to $p_b^0 \in R_+^v / \text{int } R_+^v$, or if $\|p_b^k\|$ tends to $+\infty$, then $\|{}^i b^{1k}\|$ tends to $+\infty$.

In the bonds forward markets, we show properties of ${}^i\beta_1(q_b) = \{c_b \in R^{v-1} \mid q_b \cdot c_b = 0, p_i \cdot ({}^h e_i + {}^h c_i) \geq 0, \text{ for any } p_i \in \text{supp } {}^h \psi_i(p_d^1, p_b^1, q_b)\}$, according to the results by Debreu [3]. Next, we define the i th consumer's excess demand correspondence ${}^i\zeta$ and prove the properties of ${}^i\beta_1$.

The correspondence ${}^i\beta_1(q_b)$ has the following properties:

Lemma 5.4 Correspondence ${}^i\beta_1(q_b)$ is non-empty, convex-valued, compact-valued and continuous on R_+^{v-1} .

Proof Green[3, Lemma 3.3] and Nishimura[16, Lemma 5.1]. $\square$

A rule of Grandmont's decision making is that given p^1 , for any actions ${}^i c$, ${}^i c' \in {}^i\beta_1(q_b)$, ${}^i c_b \succeq {}^i c'_b$ if and only if ${}^i v({}^i c_b, (p_d^1, p_b^1, q_b)) \geq {}^i v({}^i c'_b, (p_d^1, p_b^1, q_b))$. Therefore, maximizing ${}^i v$ on ${}^i\beta(q_b)$ yields the set $\zeta_1(q_b) = \{{}^i c_b \in R^{v-1} \mid {}^i v({}^i c_b, (p_d^1, p_b^1, q_b)) \geq {}^i v({}^i c'_b, (p_d^1, p_b^1, q_b)) \text{ for every } {}^i c'_b \}$. We call ${}^i\zeta_1(q_b)$ his demand correspondence. The demand correspondence ${}^i\zeta_1(q_b)$ has the following properties.

Lemma 5.5 ${}^i\zeta_1$ is nonempty, convex-valued, compact-valued and upper hemi-continuous. For all $q_b \in R_+^{v-1}$ and all $c_b \in {}^i\zeta_1$, $q_b \cdot {}^i c_b = 0$.

Proof Nishimura[16, Theorem 3. 6]. $\square$

The consumer's demand correspondence ${}^i\zeta_1$ satisfies a boundary condition.

Lemma 5.6 For any ${}^{ih}c_b^j \in {}^{ih}\zeta_1(q_b^k)$, if q_b^k tends to $q_b^0 \in R_+^{v-1} \setminus \text{int } R_+^{v-1}$, or if $\|q_b^k\|$ tends to $+\infty$, then $q_b^k \cdot {}^{ih}c_b^k$ tends to $+\infty$.

Proof Grandmont [5, Proposition 4. 2] □

5.2 The retired generation's optimality

Since the retired trader does not face on the uncertainty at period 1, his problem to obtain an optimal consumption at period 1 is well known. We assume that u_t is continuous, concave, bounded real-valued utility function.

Problem 5.3 Under Assumptions 4. 1 to 4. 7, at age $h(\tau + 1 \leq h \leq \tau + v - 1)$,

$$\max \sum_{t=1}^{\tau+v-h} {}^h u_t({}^h d_t),$$

$$\{{}^h d_t\}$$

$$\text{subject to } p_{d1} \cdot {}^h d_1 = {}^h b_0^0 + C^0 \cdot {}^h b^0,$$

$$p_{dt} \cdot {}^h d_t + p_{bt} \cdot {}^h b^t = p_{bt} \cdot ({}^h b^{t-1} + {}^h c_{bt}) + {}^h b_0^{t-1} + C^0 \cdot {}^h b^{t-1} \text{ for } 2 \leq t \leq v-2,$$

$$p_{dv-1} \cdot {}^h d_{v-1} = {}^h b_0^{v-2} + C^0 \cdot {}^h b^{v-2} \text{ for } t = v-1.$$

Solution Let $L_t = {}^h u_t({}^h d_t) - \lambda_t \{ p_{dt} \cdot {}^h d_t + p_{bt} \cdot {}^h b^t - p_{bt} \cdot ({}^h b^{t-1} + {}^h c_{bt}) - {}^h b_0^{t-1} - C^0 \cdot {}^h b^{t-1} \}$.

$$\partial {}^h u_t = \lambda_t \cdot p_{dt}, \quad p_{dt} \cdot {}^h d_t + p_{dt} \cdot {}^h b^t = p_{bt} \cdot ({}^h b^{t-1} + {}^h c_{bt}) + {}^h b_0^{t-1} + C^0 \cdot {}^h b^{t-1}.$$

$$\partial {}^h d_t$$

Let the solutions ${}^h d_t^*(p_{dt}), \lambda_t^*$. □

Problem 5.4 Under Assumptions 4. 1 to 4. 7, $p^1 \gg 0$, at age $h(\tau + 1 \leq h \leq \tau + v)$,

$$\max \sum_{t=2}^h \int {}^h u_t({}^h d_t^*(p_t)) d {}^h \psi_t(p_d^1, p_b^1, q_b).$$

$$\{{}^h d_t^*\}$$

Solution Since ${}^h u_t$ is concave, Jensen's inequality,

$$\sum_{t=2}^h \int {}^h u_t({}^h d_t^*(p_t)) d {}^h \psi_t(p_d^1, p_b^1, q_b) \leq \sum_{t=2}^h {}^h u_t(E[{}^h d_t^*(p_t)]).$$

Let the solutions ${}^h d_t^{**} = E[{}^h d_t^*(p_t)]$. □

6. Market Equilibrium

We shall show the existence of an equilibrium in overlapping generations' model. In an exchange economy, there are Ih consumers. In Section 3, we obtained i th consumer's demand correspondence ${}^{ih}\zeta_1$. For all $p^1 \in R_+^I$, we define the excess demand correspondence ζ as follows:

$$\zeta(p_1) = \sum_{i=1}^I [\sum_{h=1}^{\tau} \{ {}^{ih}\zeta_1(p_1) + (v/\tau) {}^{ih}am - {}^{ih}e_1 \} + \sum_{h=\tau+1}^{\tau+v} \{ {}^{ih}\zeta_1(p_1) - {}^h d_{h-1} \}] - \eta(p_d^1, p_{bv}^1).$$

Theorem 6.1 (Grandmont's Market Equilibrium Lemma)

Let $\underline{P} = \{ p_1 \in R_+^I \mid \sum_{i=1}^I p_{1i} = 1 \}$. Let D be any subset of $\underline{P}$ containing $P^0 = \{ p_1 \in \underline{P} \mid p_1 \gg 0 \}$. If each ζ^i is bounded below and ζ is a nonempty-valued correspondence whose graph is closed in $D \times R_+^I$ which satisfies:

- (1) ζ is $\square$, compact-valued and upper hemi-continuous on D ,
- (2) strong Walras' Law : $p_1 \cdot \zeta(p_1) = 0$, for all $p_1 \in D$,
- (3) a boundary condition: if p_1^k tends to $p_1^0 \in \underline{P} / D$ and for any sequence $a^k \in \zeta(p_1^k)$, there exists p_1 in D such that $p_1 \cdot a^k > 0$ for infinitely many k .

Then there exists p_1^* in D such that $0 \in \zeta(p_1^*)$.

Proof See Debreu [3, Lemma 4] and Grandmont [5, 6].

Theorem 6.2 Under Assumptions 2. 1, ..., 2. 7, there exists an equilibrium price system p_1^* .

Proof See Grandmont[5, Theorem 5. 1].

In the goods forward market, we shall show the existence of equilibrium. In an exchange economy, there are I consumers. A contract is a I -tuple $c = ({}^1c, \dots, {}^Ic) \in R^I$. $\sum_{i=1}^I (\sum_{h=1}^{\tau} {}^{ih}c + \sum_{h=\tau+1}^{\tau+\nu-1} {}^{ih}c) = 0$. An equilibrium price system is a price system q^* of P such that there exists a contract ${}^hc^* = ({}^1hc^*, \dots, {}^Ihc^*)$ with ${}^{ih}c^* \in {}^{ih}\zeta_i(q^*)$, $\sum_{i=1}^I {}^{ih}c^* = 0$ for all h .

We obtained the i th trader's demand correspondence ${}^{ih}\zeta_i$. For all $q \in P$, we define the individual excess demand correspondence ζ^i and aggregate excess demand correspondence ζ as follows:

$$\zeta(q) = \sum_{i=1}^I \sum_{h=1}^{\tau+\nu} {}^{ih}\zeta^i = \sum_{i=1}^I \sum_{h=1}^{\tau+\nu} ({}^{ih}\zeta_i(q)).$$

We assume the common expectations among all the traders. Let ${}^{ih}\Pi = \text{int co supp } {}^{ih}\psi(p_1)$ for the i th trader at age h .

Assumption 6.1 $\bigcap_{i=1}^I \bigcap_{h=1}^{\tau+\nu} {}^{ih}\Pi$ is not empty.

Theorem 6.3 Under from Assumptions 4. 1 to 4. 7, there exists an equilibrium price system q^* .

Proof See Grandmont [5, Theorem 5. 1].

We can show as the above way that there exists an equilibrium price system p_d^* and p_b^* in the bonds spot market and q_b^* in the forward market.

References

1. P. Billingsley, "Convergence of Probability Measures," John Wiley, 1969.
2. D. Cass and M. E. Yari, A Reexamination of the Pure Consumption Loans Model, Journal of Political Economics 74, 1966, 353-367.
3. G. Debreu, Existence of competitive equilibrium, in K. J. Arrow and M. D. Inriligator (eds.) "Handbook of Mathematical Economics, vol. II," North-Holland, Amsterdam, 1982.
4. J.M. Grandmont, Continuity properties of von Neumann-Morgenstern utility, Journal of Economic Theory 4, 1972, pp47-57.
5. -----, On the Short Run Equilibrium in a Monetary Economy, in J. Drèze (ed.) "Allocation under

- Uncertainty, Equilibrium, and Optimality,” Macmillan & Co., London, 1974, pp. 213-228.
6. J.M. Grandmont and W. Hildenbrand, Stochastic Processes of Temporary Equilibria, *Journal of Mathematical Economics* 1, 1974, pp. 247-277.
 7. -----, Temporary general equilibrium theory, in K.J. Arrow and M. D. Intriligator (eds.) “Handbook of Mathematical Economics, vol. II,” North-Holland, Amsterdam, 1982.
 8. -----, “Money and value,” *Maison des Sciences de l’Homme* and Cambridge University Press, 1983.
 9. J.M. Grandmont and G. Laroque, Money in the Pure Consumption Loan Model, *Journal of Economic Theory* 6, 1973, pp.382-397.
 10. J.M. Grandmont and Y. Younes, On the Role of Money and the Existence of a Monetary Equilibrium, *Review of Economic Studies* 39, 1972, pp.355-372.
 11. -----, On the Efficiency of a Monetary Equilibrium, *Review of Economic Studies* 40, 1973, pp.149-165.
 12. J. R. Hicks, “Value and Capital,” 2 nd edn., Clarendon Press, Oxford, 1946.
 13. H. Nikaido, “Convex structures and economic theory,” Academic Press, New York, 1968.
 14. K. Nishimura, Multivariate von Neumann-Morgenstern Utility and its Additive Representation, *Otemon Economic Studies* 16, 1983, pp. 1-14.
 15. -----, Temporary General Equilibrium in a Monetary Economy with Multi-Planning Periods, in *Essays and Studies published in Commemoration of the 20th Anniversary of Otemon-Gakuin University, Faculty of Economics*, 1987, pp.157-169, in Japanese.
 16. -----, Temporary General Equilibrium in Overlapping Generations Model, 33, 2000, pp. 59-68.
 17. -----, Temporary General Equilibrium in Spot and Forward Markets with Fiat Money, *Otemon Economic Studies* 44, 2011, pp. 27-35.
 18. P. A. Samuelson, An Exact Consumption-loan Model of Interest with or without the Social Contrivance of Money, *Journal of Political Economy* 66 (1958), 467-482.
 19. D. Sonderman, Temporary Competitive Equilibrium under Uncertainty, in J. Drèze (ed.) “Allocation under Uncertainty, Equilibrium, and Optimality,” Macmillan & Co., London, 1974, pp. 229-256.

(Received 21. Dec., 2015)